# Anderson Localization

## ( and Intrinsic localized modes of phonons)

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Izmir July 2017

Recent Progress in the Physics of Thermal Transport - ICTP-ECAR

# Outline

Introduction

Basic concepts of the theory of one-particle localization (Andrson Model)

Brief review of the Weak Localization

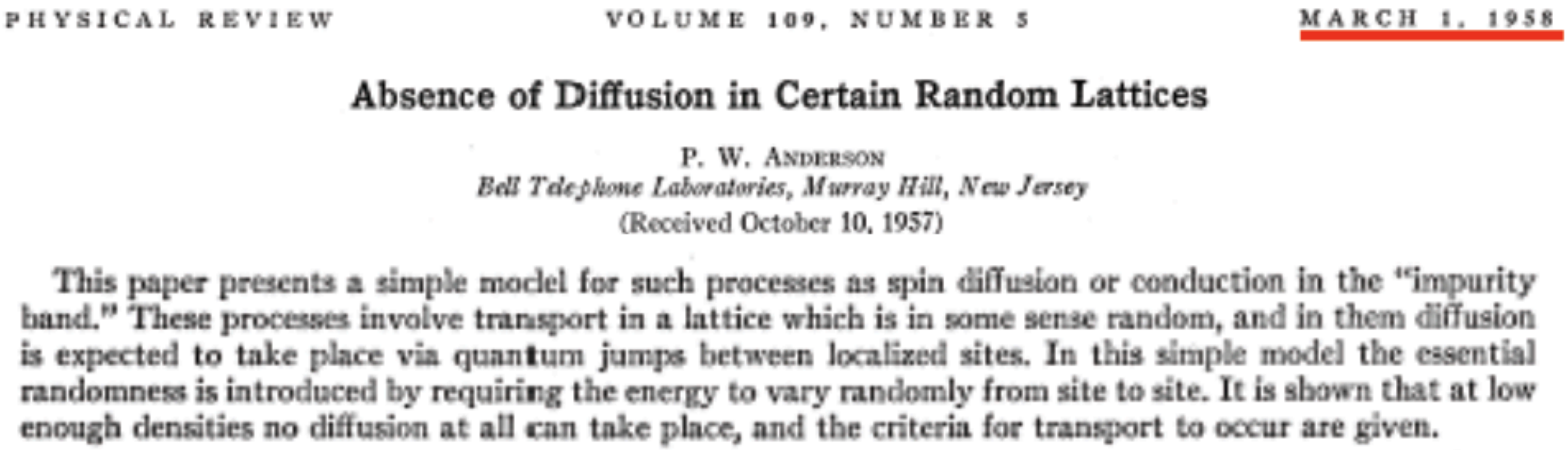
Spectral and LDOS Statistics

Localization Length

Non-Linear Wave Equation and Phonon Localization (Intrinsic localized modes of phonons)

- See Altshuler's lecture notes and Binninger's notes
- **Disorder and interference: localization phenomena, C.A. Muller and**

# Almost 60 years of Anderson Localization

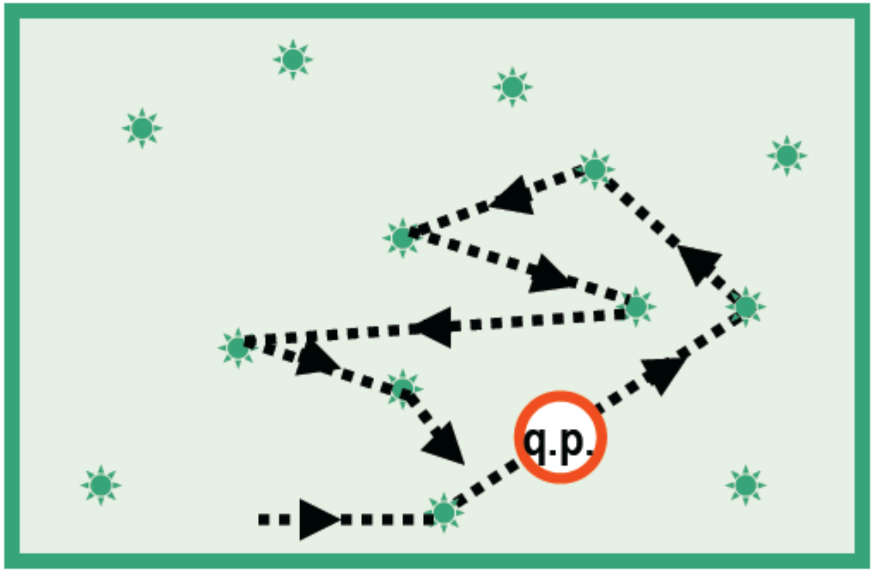


PHYSICAL REVIEW VOLUME 109, NUMBER 5 MARCH 1, 1958

## Absence of Diffusion in Certain Random Lattices

P. W. ANDERSON
*Bell Telephone Laboratories, Murray Hill, New Jersey*
(Received October 10, 1957)

This paper presents a simple model for such processes as spin diffusion or conduction in the "impurity band." These processes involve transport in a lattice which is in some sense random, and in them diffusion is expected to take place via quantum jumps between localized sites. In this simple model the essential randomness is introduced by requiring the energy to vary randomly from site to site. It is shown that at low enough densities no diffusion at all can take place, and the criteria for transport to occur are given.

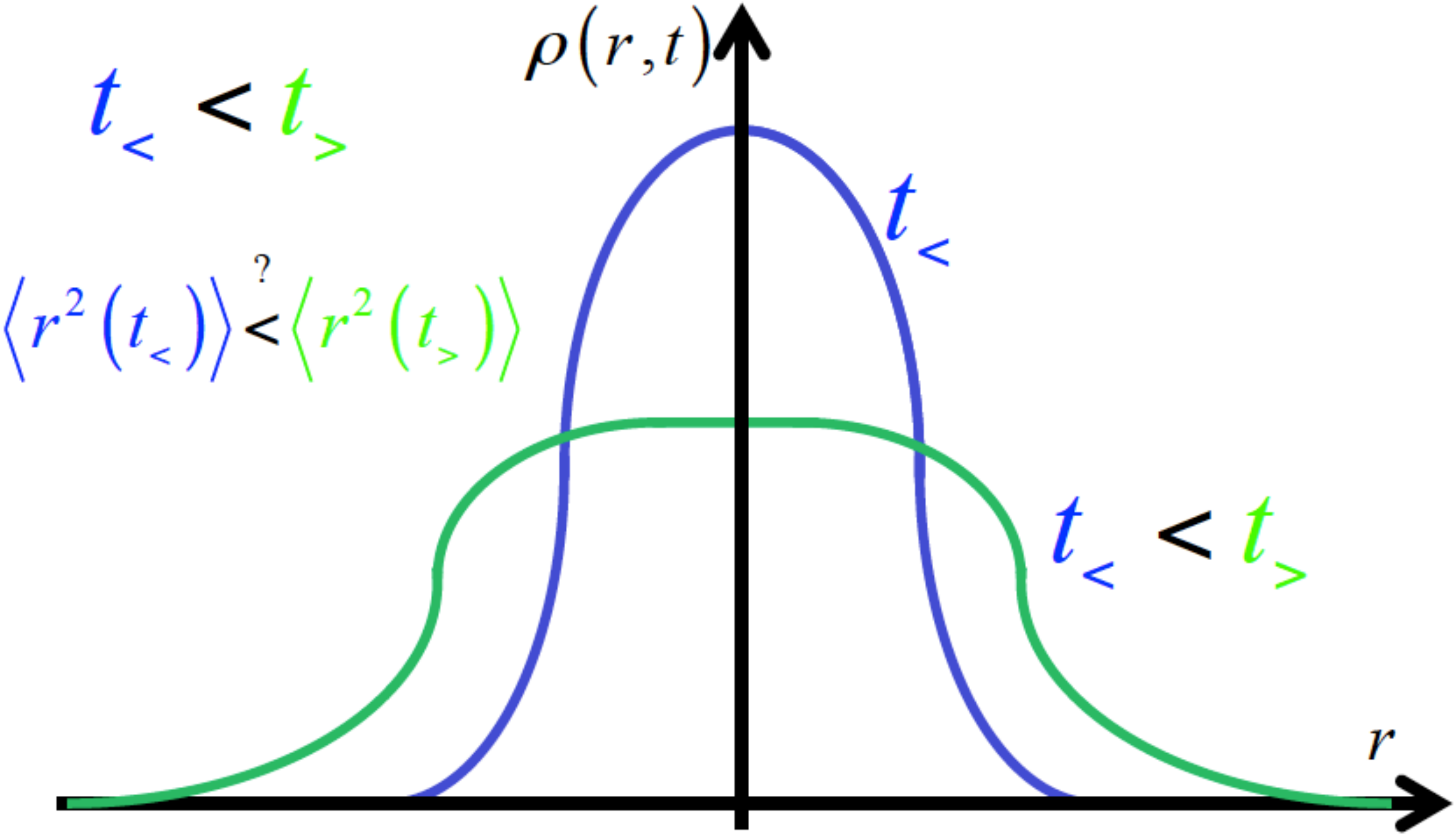


…very few believed it [localization] at the time, and even fewer saw its importance; among those who failed to fully understand it at first was certainly its author…

Nobel Lecture

Nobel Lecture, December 8, 1977

Local Moments and Localized States



$t_< < t_>$

$$\langle r^2(t_<)\rangle \overset{?}{<} \langle r^2(t_>)\rangle$$

How does a density fluctuation (wave packet) spread ?

# Diffusion Equation

$$\frac{\partial \rho}{\partial t} - D\nabla^2 \rho = 0$$

$$\langle r^2 \rangle = Dt$$

Diffusion constant

$\rho(r,t)$ Can be density of particles or energy density. It can also be the **probability** to find a particle at a given point at a given time

**Einstein theory of Brownian motion, 1905**

The diffusion equation is valid for any random walk provided that there is no memory (markovian process)

## Einstein (1905):

$$\left\langle r^2 \right\rangle = Dt$$

Random walk

⇩

always diffusion

as long as the system
has no memory

## Anderson(1958):

For quantum
particles

⇩

not always!

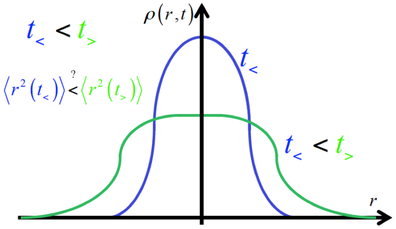
It might be that

$$\left\langle r^2 \right\rangle \xrightarrow[t\to\infty]{} const$$

$$D = 0$$

Quantum interference ⇨ memory

Why ?



# Basic Quantum Mechanics:

$$\left[-\frac{\nabla^2}{2m}+U(\boldsymbol{r})-\epsilon_F\right]\psi_\alpha(\boldsymbol{r})=\xi_\alpha\psi_\alpha(\boldsymbol{r})$$

$\xi_\alpha$ -spectrum

Random Potential

**Spectra**

**Continuous**

**Unbound states**

$$\left|\psi_\alpha\left(\vec{r}\right)\right|^2\xrightarrow[L\to\infty]{}O\left(L^{-d}\right)$$

**Discrete**

**Bound states**

$$\left|\psi_\alpha\left(\vec{r}\right)\right|^2\xrightarrow[\left|\vec{r}\right|\to\infty]{}O\left(e^{-\left|\vec{r}\right|/\varsigma}\right)$$

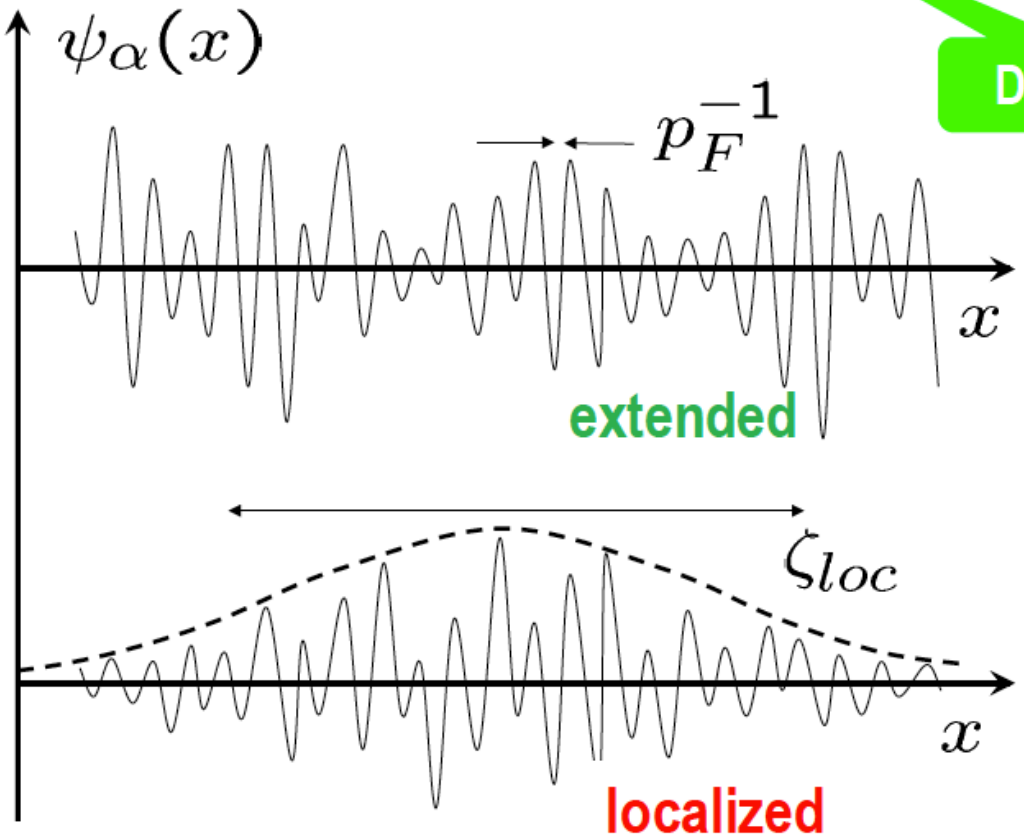
$L$ System size

$d$ Number of the spatial dimensions

**Localization of one-particle wave-functions:**

$$\left[-\frac{1}{2m}\nabla^2+U(\vec{r})\right]\psi_\alpha(\vec{r})=E_\alpha\psi_\alpha(\vec{r})$$

Disorder



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$$\mathit{Critical}\quad \mathit{states}\rightarrow\psi_\alpha(x)\sim\frac{1}{|x-x_0|^\gamma}$$

# Einstein Relation (1905)

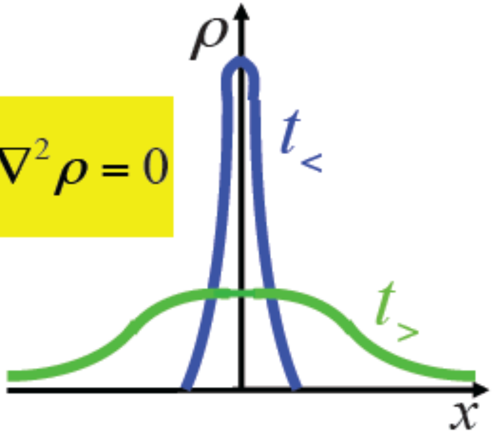
$$\sigma = e^2 D \nu \qquad \nu \equiv \frac{dn}{d\mu}$$

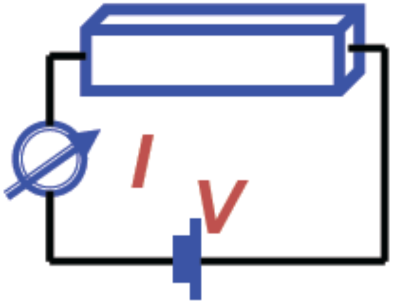
**Conductivity**

**Diffusion Constant**

**Density of states**

$$\frac{\partial \rho}{\partial t} - D\nabla^2 \rho = 0$$





$$G = \left(\frac{I}{V}\right)_{V=0}$$

$$\sigma = G\frac{L}{A}$$

Extended states:

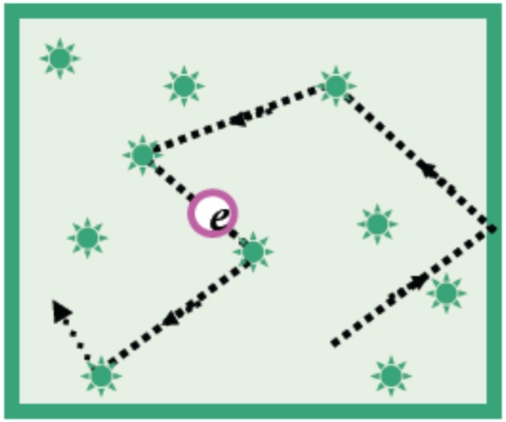
$$G \propto L^{d-2};$$

$$\sigma \xrightarrow[L\to\infty]{} const; \quad D \xrightarrow[L\to\infty]{} const$$

Localized states:

$$G \propto e^{-L/\xi}; \quad \sigma = 0; \quad D = 0$$

# Anderson Model



✸ *Scattering centers,*
*e.g., impurities*

## Models of disorder:

Randomly located impurities

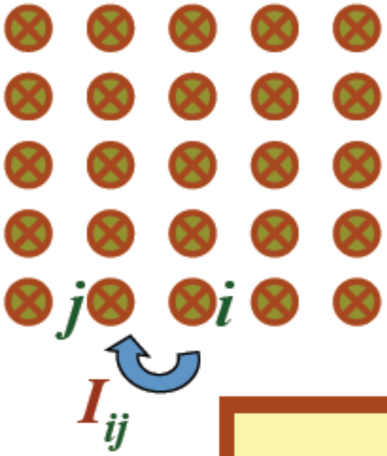
White noise potential

Lattice models

- Anderson model
- Lifshits model

**Noninteracting electrons**

# Anderson Model



- *Lattice - tight binding model*
- *Onsite energies* $\varepsilon_i$ - *random*
- *Hopping matrix elements* $I_{ij}$

$$-W < \varepsilon_i < W$$

*uniformly distributed*

$$I_{ij} = \begin{cases} I & i \text{ and } j \text{ are nearest neighbors} \\ 0 & \text{otherwise} \end{cases}$$

## Anderson Transition

$$I_c = f(d) * W$$

$$I < I_c$$

**Insulator**

*All eigenstates are localized*

*Localization length* $\zeta$

$$I > I_c$$

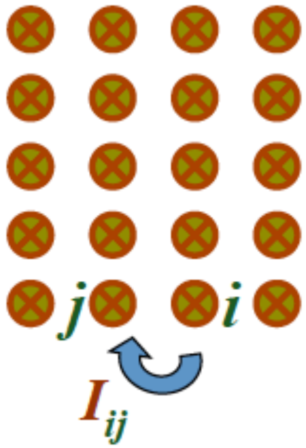
**Metal**

*There appear states extended all over the whole system*

## One-dimensional Anderson Model

$$\hat{H} = \begin{pmatrix} \varepsilon_1 & I & & 0 \\ & & 0 & \\ I & 0 & & I \\ & 0 & & \\ 0 & I & & \varepsilon_N \end{pmatrix}$$

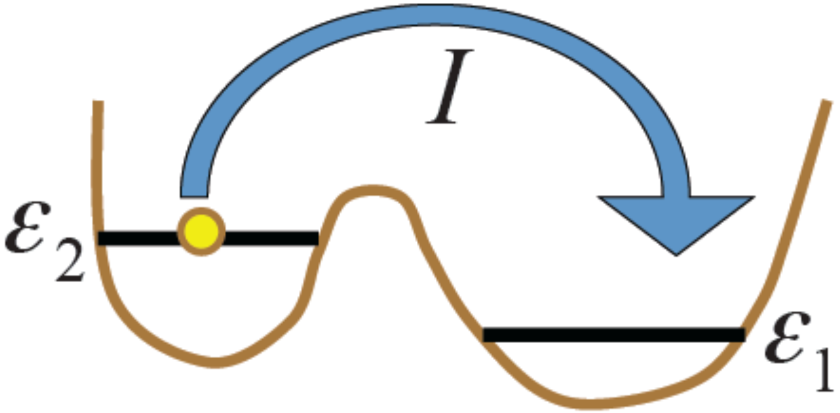
Q: Why arbitrary weak hopping $I$ is not sufficient for the existence of the diffusion ?



**Einstein (1905):** Marcovian (no memory) process → diffusion

Quantum mechanics is not marcovian
There is memory in quantum propagation !

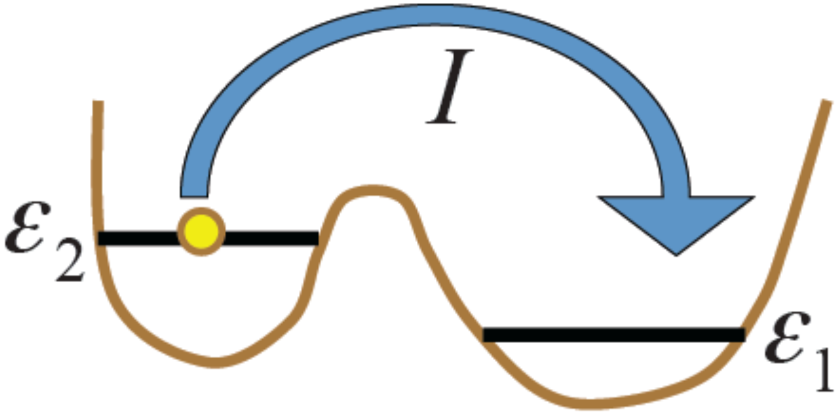
Why ?



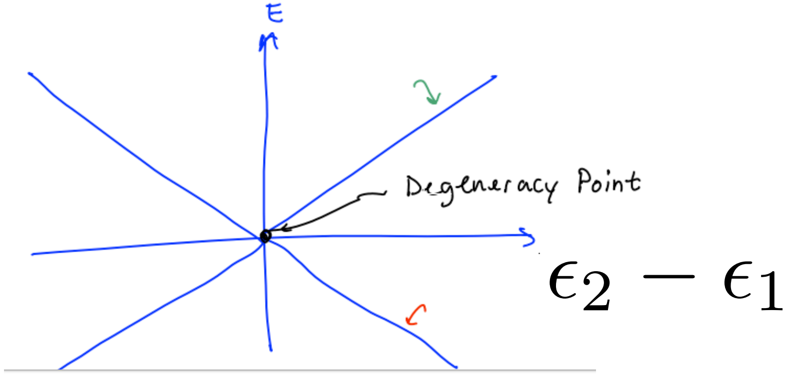
Hamiltonian

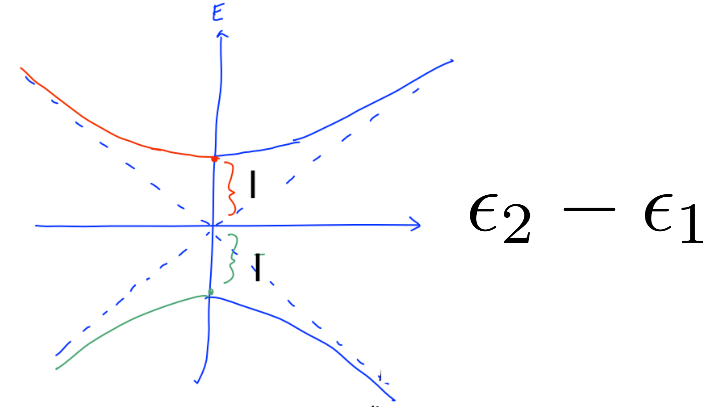
$$\hat{H} = \begin{pmatrix} \varepsilon_1 & I \\ I & \varepsilon_2 \end{pmatrix} \xrightarrow{\text{diagonalize}} \hat{H} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

$$E_2 - E_1 = \sqrt{(\varepsilon_2 - \varepsilon_1)^2 + I^2}$$



Hamiltonian





$$\hat{H} = \begin{pmatrix} \varepsilon_1 & I \\ I & \varepsilon_2 \end{pmatrix} \xrightarrow{\text{diagonalize}} \hat{H} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

$$\boxed{E_2 - E_1 = \sqrt{(\varepsilon_2 - \varepsilon_1)^2 + I^2}}$$

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$$E_2 - E_1 = \sqrt{\left(\varepsilon_2 - \varepsilon_1\right)^2 + I^2} \approx \begin{matrix} \varepsilon_2 - \varepsilon_1 & \varepsilon_2 - \varepsilon_1 >> I \\ I & \varepsilon_2 - \varepsilon_1 << I \end{matrix}$$

von Neumann & Wigner “noncrossing rule”

Level repulsion

*v. Neumann J. & Wigner E. 1929 Phys. Zeit. v.30, p.467*

**What about the eigenfunctions ?**

$$\hat{H} = \begin{pmatrix} \varepsilon_1 & I \\ I & \varepsilon_2 \end{pmatrix}$$

$$E_2 - E_1 = \sqrt{\left(\varepsilon_2 - \varepsilon_1\right)^2 + I^2} \approx \begin{matrix} \varepsilon_2 - \varepsilon_1 & \varepsilon_2 - \varepsilon_1 >> I \\ I & \varepsilon_2 - \varepsilon_1 << I \end{matrix}$$

# What about the eigenfunctions ?

$$\phi_1, \varepsilon_1; \phi_2, \varepsilon_2 \quad \Leftarrow \quad \psi_1, E_1; \psi_2, E_2$$

$$\varepsilon_2 - \varepsilon_1 >> I$$

$$\psi_{1,2} = \varphi_{1,2} + O\left(\frac{I}{\varepsilon_2 - \varepsilon_1}\right)\varphi_{2,1}$$

**Off-resonance**

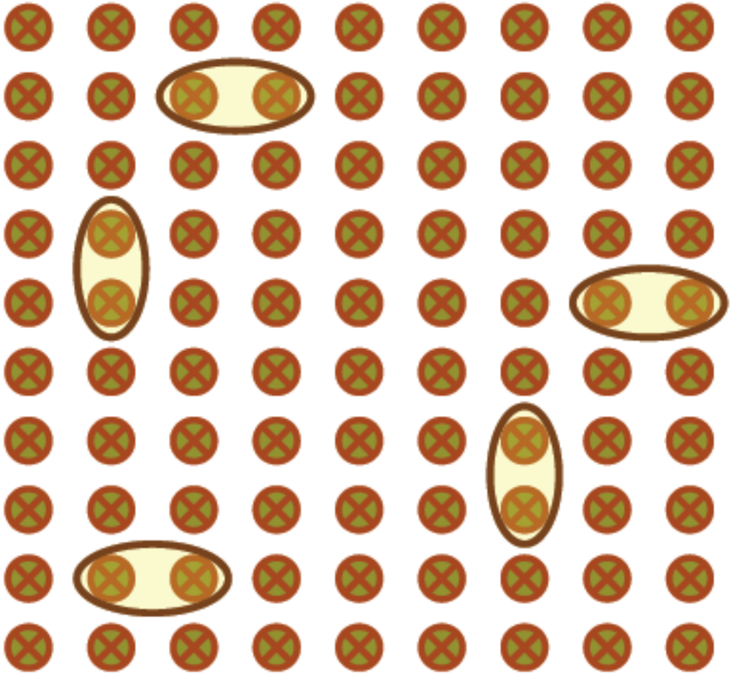
**Eigenfunctions are close to the original on-site wave functions**

$$\varepsilon_2 - \varepsilon_1 << I$$

$$\psi_{1,2} \approx \varphi_{1,2} \pm \varphi_{2,1}$$

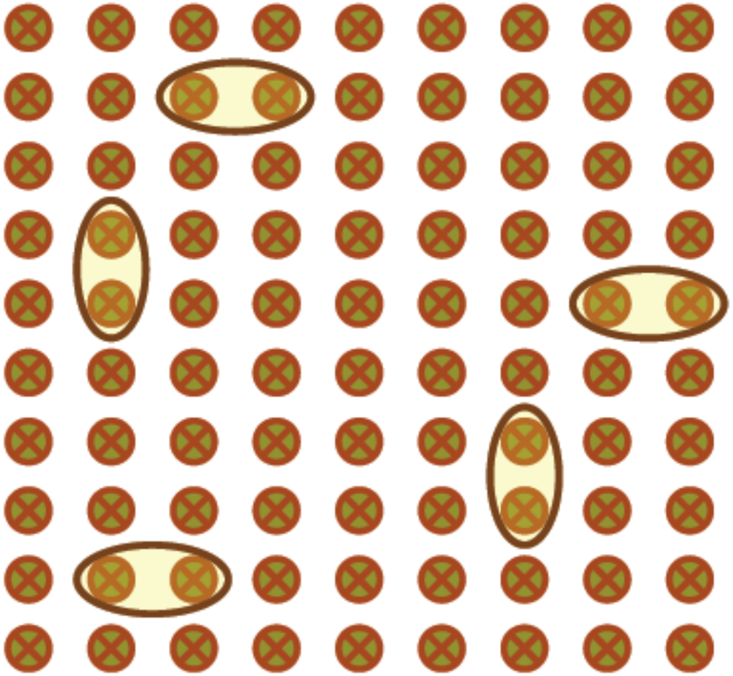
**Resonance**

**In both eigenstates the probability is equally shared between the sites**



**Anderson insulator**

Few isolated resonances



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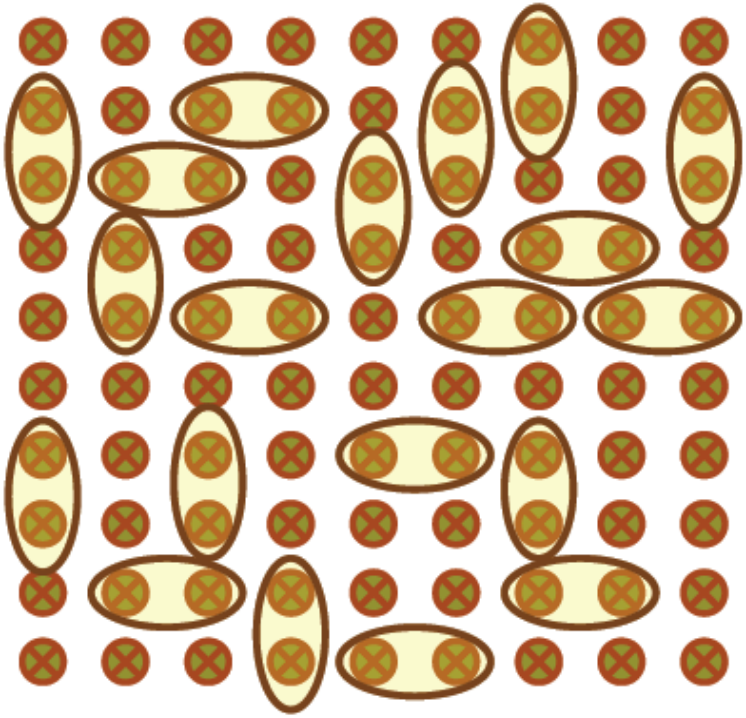


**Anderson metal**

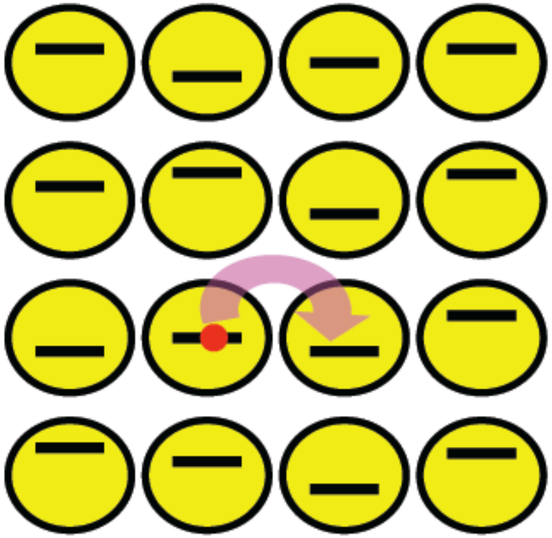
There are many resonances
and they overlap

Quantum percolation!

# Anderson Transition

# Anderson Model

- one particle,
- one level per site,
- onsite disorder
- nearest neighbor hoping



**Basis:** $\left|i\right\rangle$, $i$ **labels sites**

**Hamiltonian – matrix with random diagonal:**

$$\hat{H} = \hat{H}_0 + \hat{V} \qquad \hat{H}_0 = \sum_i \varepsilon_i \left|i\right\rangle\left\langle i\right| \qquad \hat{V} = \sum_{i,j=n.n.} I\left|i\right\rangle\left\langle j\right|$$

$-W \le \varepsilon_i \le W$ **– random**

# Anderson's recipe:

Consider a closed finite system: $N < \infty$ number of sites

number of quantum states $E_\alpha, \psi_\alpha(i)$

## Global density of states

$$\nu(E) \equiv N^{-1} \sum_{\alpha=1}^{N} \delta(E - E_\alpha)$$

Broadening: $\boxed{\delta(E - E_i) \Leftarrow \delta_\eta(E - E_i) = \frac{1}{\pi} \operatorname{Im} \frac{1}{E - E_i - i\eta}}$

Limits: $N \to \infty$ first

$\eta \to 0$ afterwards

⇨ Global density of states $\nu(E)$ becomes a continuous and smooth function

## Local density of states

$$\nu(E,i) \equiv \sum_{\alpha=1}^{N} \delta(E - E_\alpha) |\psi_\alpha(i)|^2 \equiv \frac{1}{\pi} \operatorname{Im} G_{ii}(E)$$

After taking the limits:

$I = 0$ $|\psi_\alpha(i)|^2 = \delta_{\alpha,i}\,, E_\alpha = \varepsilon_i$ $\operatorname{Im} G_{ii}(E) = \pi\delta(E - \varepsilon_i)$ set of delta-functions

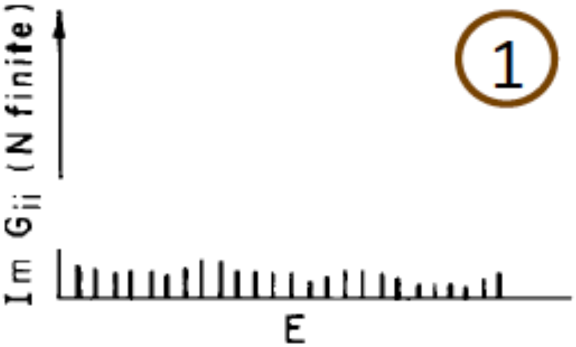
$W = 0$ Local density of states is a continuous and smooth function

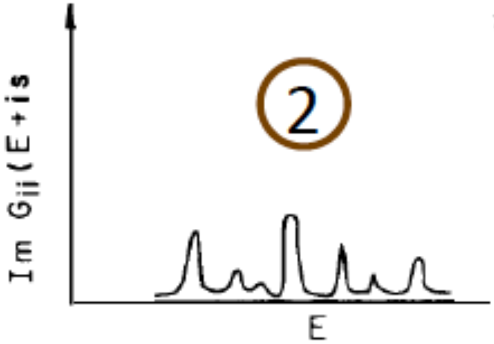
$$\rho(E) = (1/N)\sum_{n=1}^{N}\delta(E-\lambda_n) = \lim_{\eta\to 0^+}(1/N\pi)\sum_{i=1}^{N}\Im G_{ii}$$

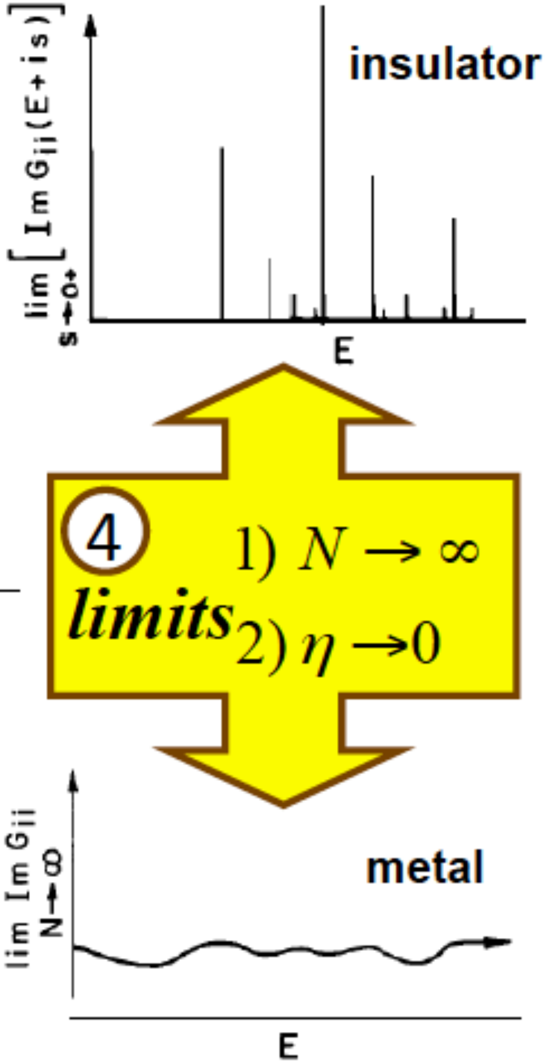
$$\text{(IPR)}\ \langle\Upsilon_{2,n}\rangle = \langle\sum_{i=1}^{N}|\langle i|n\rangle|^4\rangle = \lim_{\eta\to 0^+}(1/N)\sum_{i=1}^{N}\eta|G_{ii}|^2.$$

# Anderson's recipe:

1. take discrete spectrum $\varepsilon_i$ of $H_0$
2. Add an infinitesimal *Im* part $i\eta$ to $\varepsilon_i$
3. Evaluate $Im\Sigma_i$ imaginary part of the renormalized energy



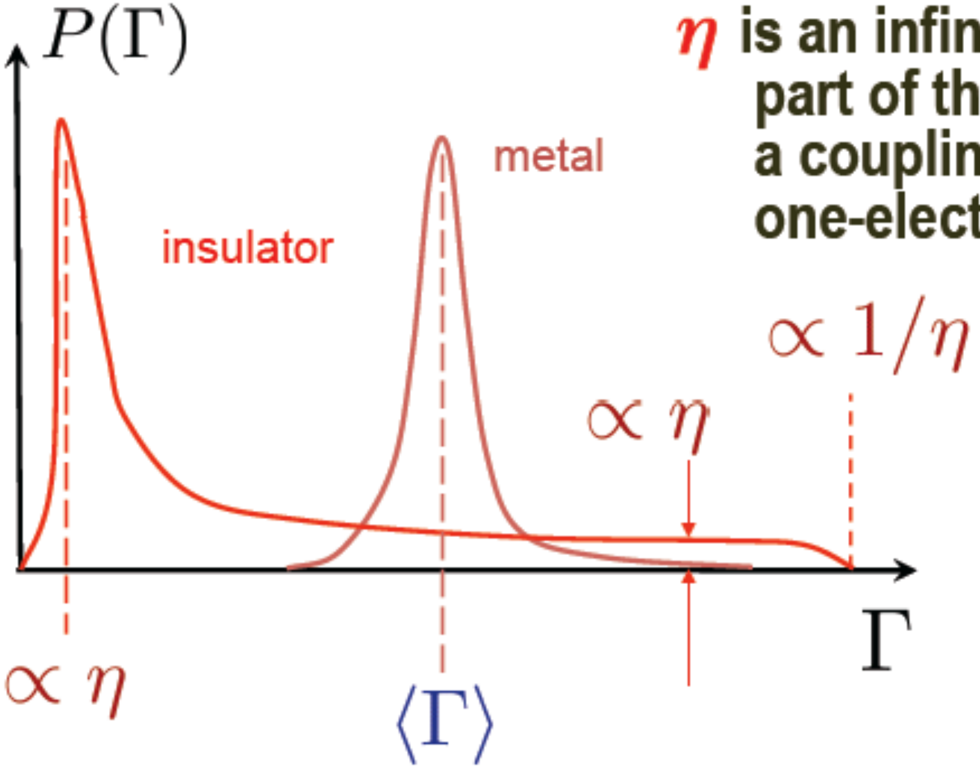




4. take limit $\eta \to 0$ but only after $N \to \infty$
5. "What we really need to know is the *probability distribution* of $Im\Sigma_i$ *not* its average…" P.W. Anderson Nobel Lecture

!

# Probability Distribution of $\Gamma = Im\ \Sigma$

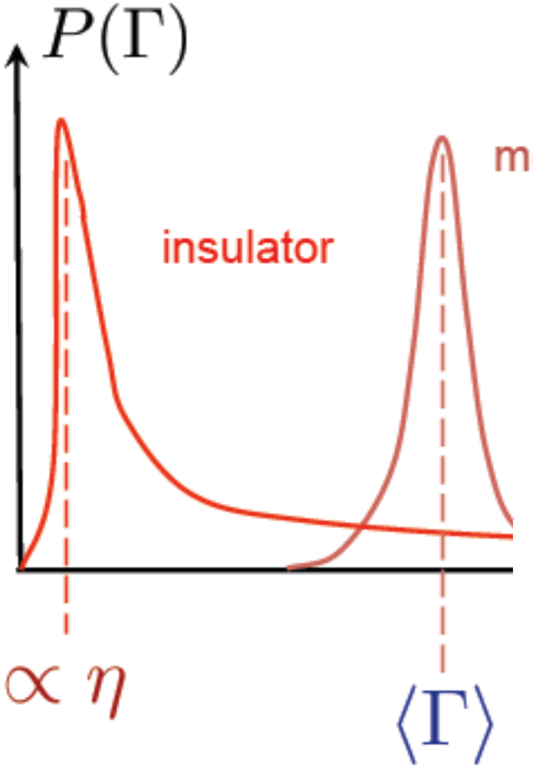


$\eta$ is an infinitesimal width (***Im*** part of the self-energy due to a coupling with a bath) of one-electron eigenstates

***Look for:***

$$\lim_{\eta \to +0} \lim_{V \to \infty} P(\Gamma > 0) = \begin{cases} > 0; & metal \\ 0; & insulator \end{cases}$$

# Probability Distribution of *Γ=Im Σ*



***η*** is an infinitesimal width (***Im*** part of the self-energy due to a coupling with a bath) of one-electron eigenstates

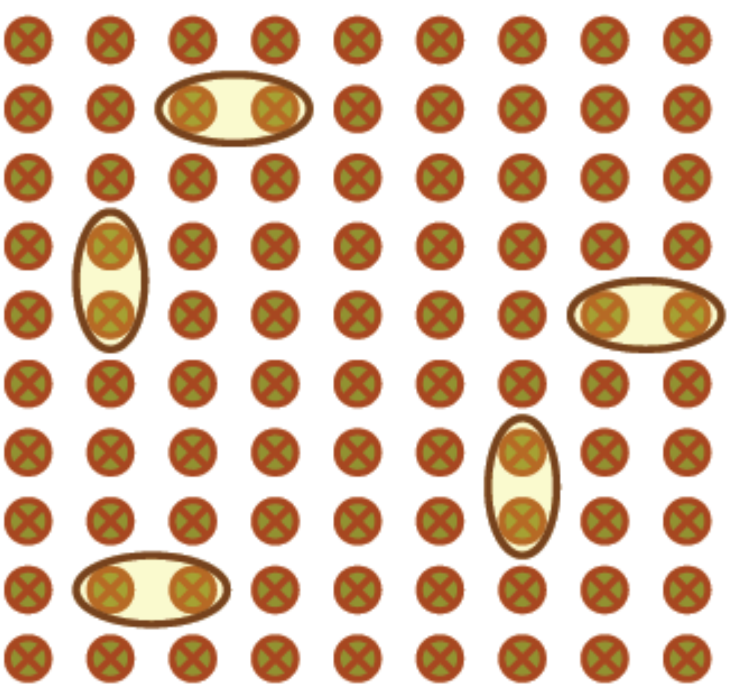
$\propto 1/\eta$

$\propto \eta$

$$\lim_{s\to 0} P\left(\mathrm{Im}\left(\frac{\Sigma}{s}\right)=X\right)\mathrm{d}X=\frac{\mathrm{d}X}{X^{3/2}}\,\mathrm{e}^{-\frac{s}{X}}$$

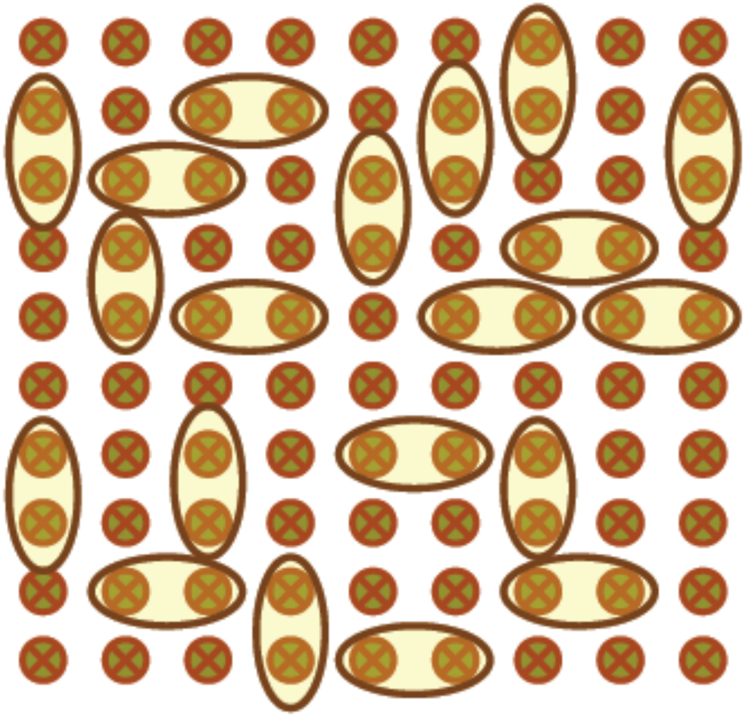
***Look for:***

$$\lim_{\eta\to+0}\ \lim_{V\to\infty} P(\Gamma>0)=\begin{cases} >0; & \mathit{metal} \\ 0; & \mathit{insulator}\end{cases}$$



Anderson insulator

Few isolated resonances

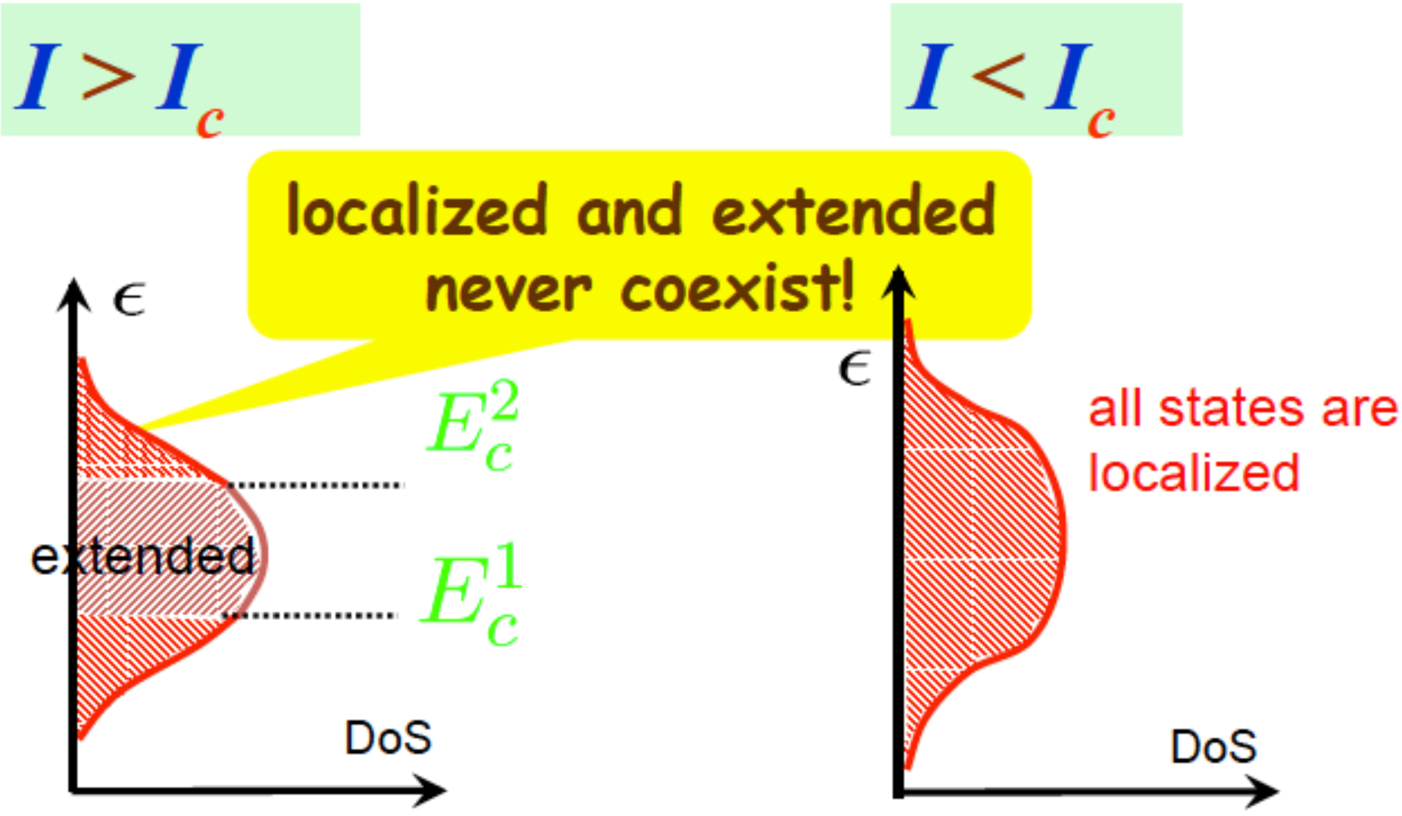


Anderson metal

There are many resonances and they overlap

**Transition:** Typically each site is in the resonance with some other one

# Anderson Transition



$I > I_c$

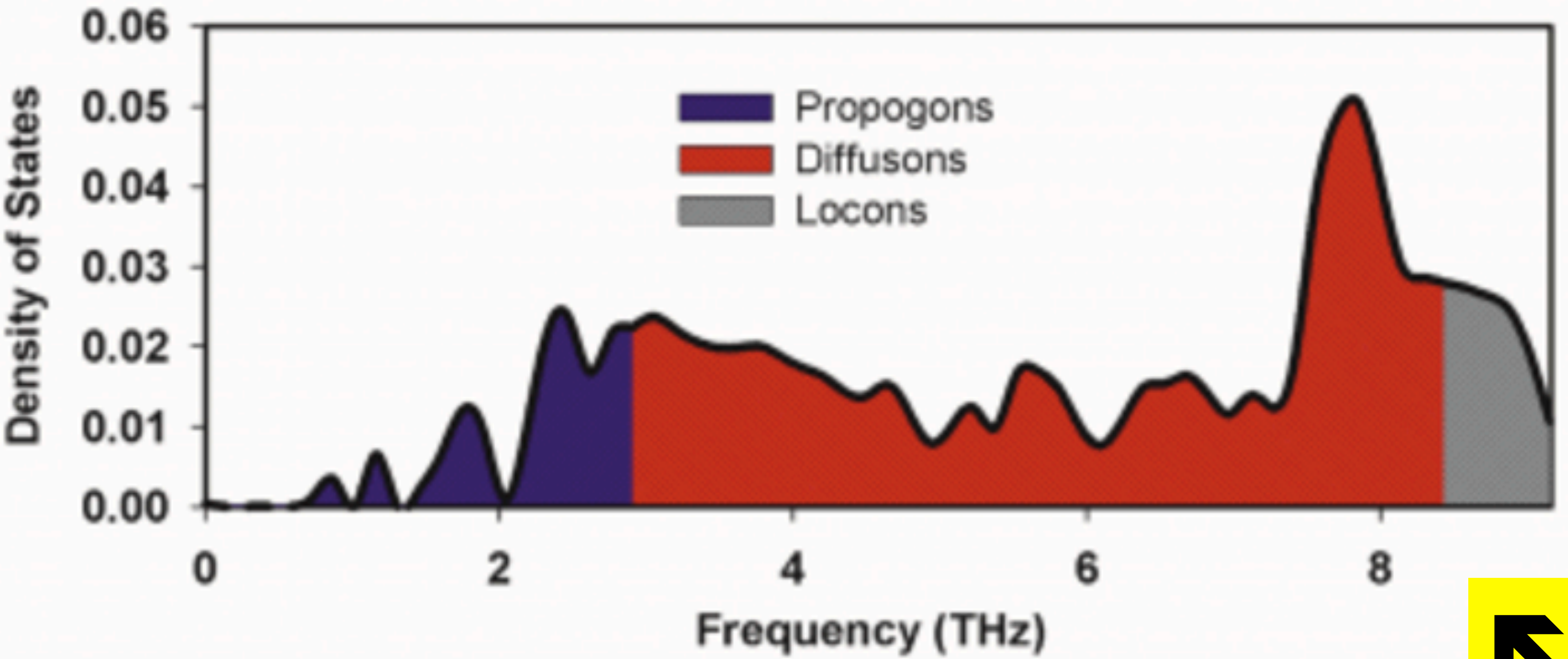
$I < I_c$

localized and extended never coexist!

$E_c$ - mobility edges (one particle)

Tarquini et al. PRL 116, 010601 (2016)

Equation for mobility edge in terms of disorder properties



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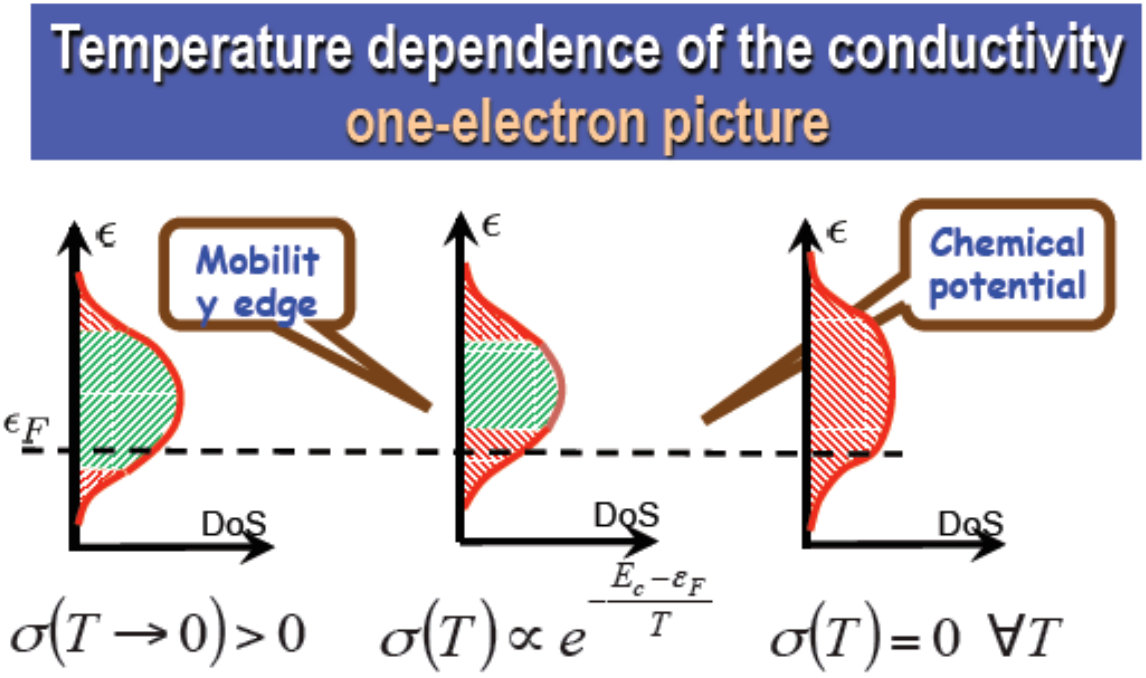
Tarquini et al. PRL 116, 010601 (2016)
Equation for mobility edge in terms of disorder properties



## Temperature dependence of the conductivity
### one-electron picture

$$\sigma(T\to 0)>0 \qquad \sigma(T)\propto e^{-\frac{E_c-\varepsilon_F}{T}} \qquad \sigma(T)=0\ \ \forall T$$

$$I > I_c \qquad\qquad\qquad I < I_c$$

There are extended states

All states are localized

# LOCALIZATION TRANSITION IN THE ANDERSON MODEL ON THE BETHE LATTICE: SPONTANEOUS SYMMETRY BREAKING AND CORRELATION FUNCTIONS

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Received 21 June 1991

We present the complete analytical solution of the Anderson model on the Bethe lattice. Within the scope of the supersymmetric approach the delocalization transition manifests itself as a spontaneous breaking of the UOSP(2, 2/2, 2) invariance and can be described by means of the order-parameter function. We attribute a clear physical meaning to this function providing the explicit connection with the known behaviour of Green functions in disordered systems. Apart from reproducing the known results for the position of the mobility edge, we calculate the density–density correlation function in both localized and extended phases. The found critical behaviour contradicts to the one-parameter scaling hypothesis, in agreement with results obtained in the framework of the supermatrix $\sigma$-model on the Bethe lattice.

Nuclear Physics B366 (1991) 507–532
North-Holland

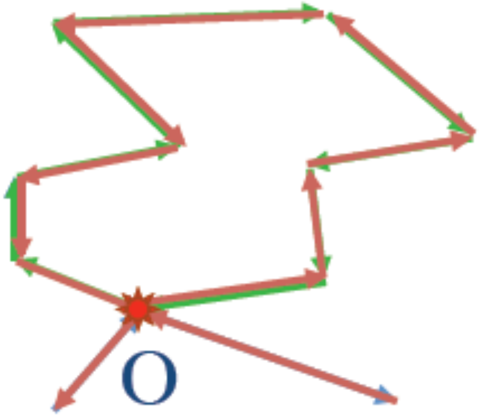
NUCLEAR
PHYSICS B

# Low dimensions: weak localization

# WEAK LOCALIZATION

$$\phi = \int p dr$$

Phase accumulated
when traveling
along the loop



The particle
can go around
the loop in
two directions

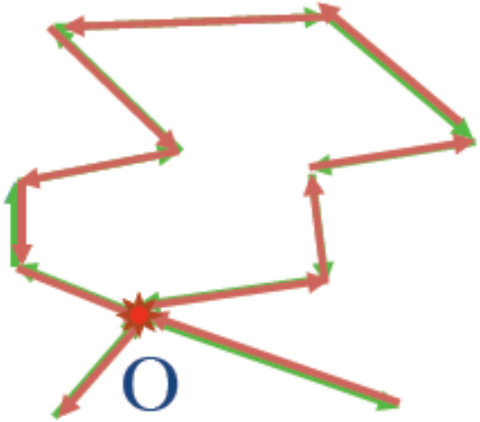
**Memory!**

$$\varphi_1 = \varphi_2$$

Constructive interference → probability to return to the origin gets enhanced → quantum corrections reduce the diffusion constant.

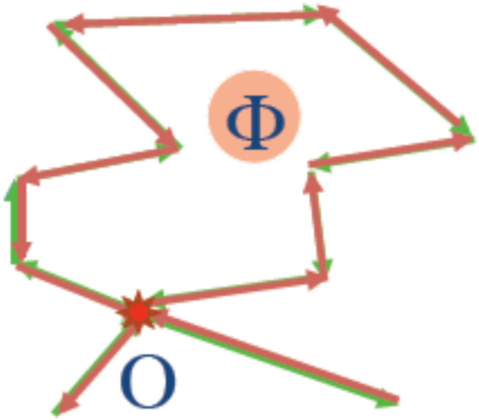
**Tendency towards localization**

# Magnetoresistance



*No magnetic field*

$$\varphi_1 = \varphi_2$$



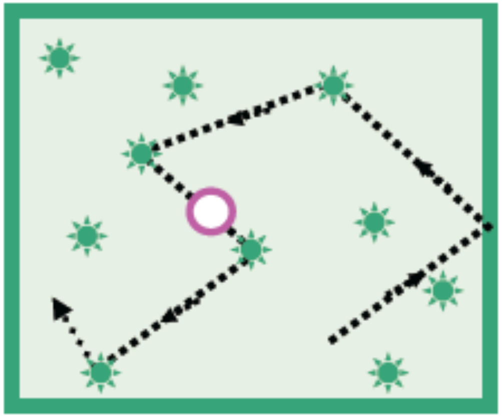
*With magnetic field* $H$

$$\varphi_1 - \varphi_2 = 2*2\pi\,\Phi/\Phi_0$$

# One parameter scaling theory of Anderson localization

# Classical particle in a random potential

# Diffusion



1 particle – random walk

Density of the particles $\rho$

Density fluctuations $\rho(\mathbf{r},t)$ at a given point in space $\mathbf{r}$ and time $t$.

$$\frac{\partial\rho}{\partial t} - D\nabla^2\rho = 0$$

Diffusion Equation

$D$ - Diffusion constant

$$D = \frac{l^2}{d\tau}$$

$l$ mean free path

$\tau$ mean free time

$d$ # of dimensions

$$\frac{1}{l} = s n_{im}$$

Cross-section

Impurity concentration

# Thouless Conductance

**Einstein Relation for electric conductivity $\sigma$**

$$\sigma = e^2 D\nu \qquad \nu \equiv \frac{dn}{d\mu}$$

**Conductance**

$$G = \sigma L^{d-2}$$

for a cubic sample of the size $L$

$$G = \frac{e^2}{h}(\nu L^d)\frac{Dh}{L^2} = \frac{e^2}{h}g(L)$$

$$\boxed{g(L) = \frac{hD/L^2}{1/\nu L^d}} = \frac{\text{Thouless energy}}{\text{mean level spacing}}$$

**Dimensionless Thouless conductance**

## Thouless Conductance

$$G = \frac{e^2}{h} g(L)$$

$$g(L) \equiv \frac{\text{Thouless energy}}{\text{mean level spacing}}$$

Dimensionless
Thouless conductance

## Thouless Energy

$$E_T \equiv \frac{hD}{L^2}$$ : Inverse escape time

Extra energy scale as compared with the generic Random Matrix theory ⇒ Dimensionless parameter

$$g(L) = \frac{hD/L^2}{1/\nu L^d} = \frac{E_T}{\delta_1} = \frac{\text{Thouless energy}}{\text{mean level spacing}}$$

Thouless conductance is Dimensionless

Corrections to the diffusion come from the large distances (infrared corrections)

**Scaling theory of Localization**

(Abrahams, Anderson, Licciardello and Ramakrishnan, 1979)

**Universal description!**

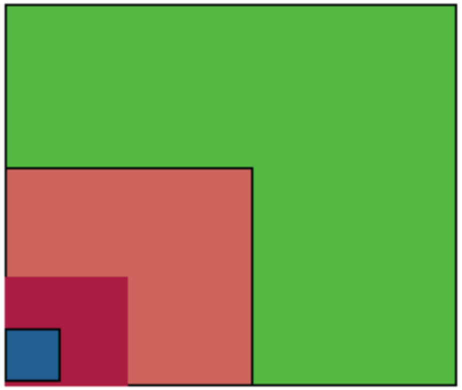
# Scaling theory of Localization

**Abrahams, Anderson, Licciardello and Ramakrishnan 1979**

$g = E_T / \delta_1$

Dimensionless Thouless conductance

$g = Gh/e^2$



$$L = 2L = 4L = 8L \ldots.$$

without quantum corrections

$$E_T \propto L^{-2} \qquad \delta_1 \propto L^{-d}$$

$${E_T \atop \delta_1} \longrightarrow {E_T \atop \delta_1} \longrightarrow {E_T \atop \delta_1} \longrightarrow {E_T \atop \delta_1}$$

$$g \longrightarrow g \longrightarrow g \longrightarrow g$$

$$\boxed{\frac{d(\log g)}{d(\log L)} = \beta(g)}$$

$$\boxed{\frac{d(\log g)}{d(\log L)} = \beta(g)}$$

# $\beta -$ function

Is universal, i.e., material independent

But

It depends on the global symmetries, e.g., it is different with and without $T$-invariance

## Limits:

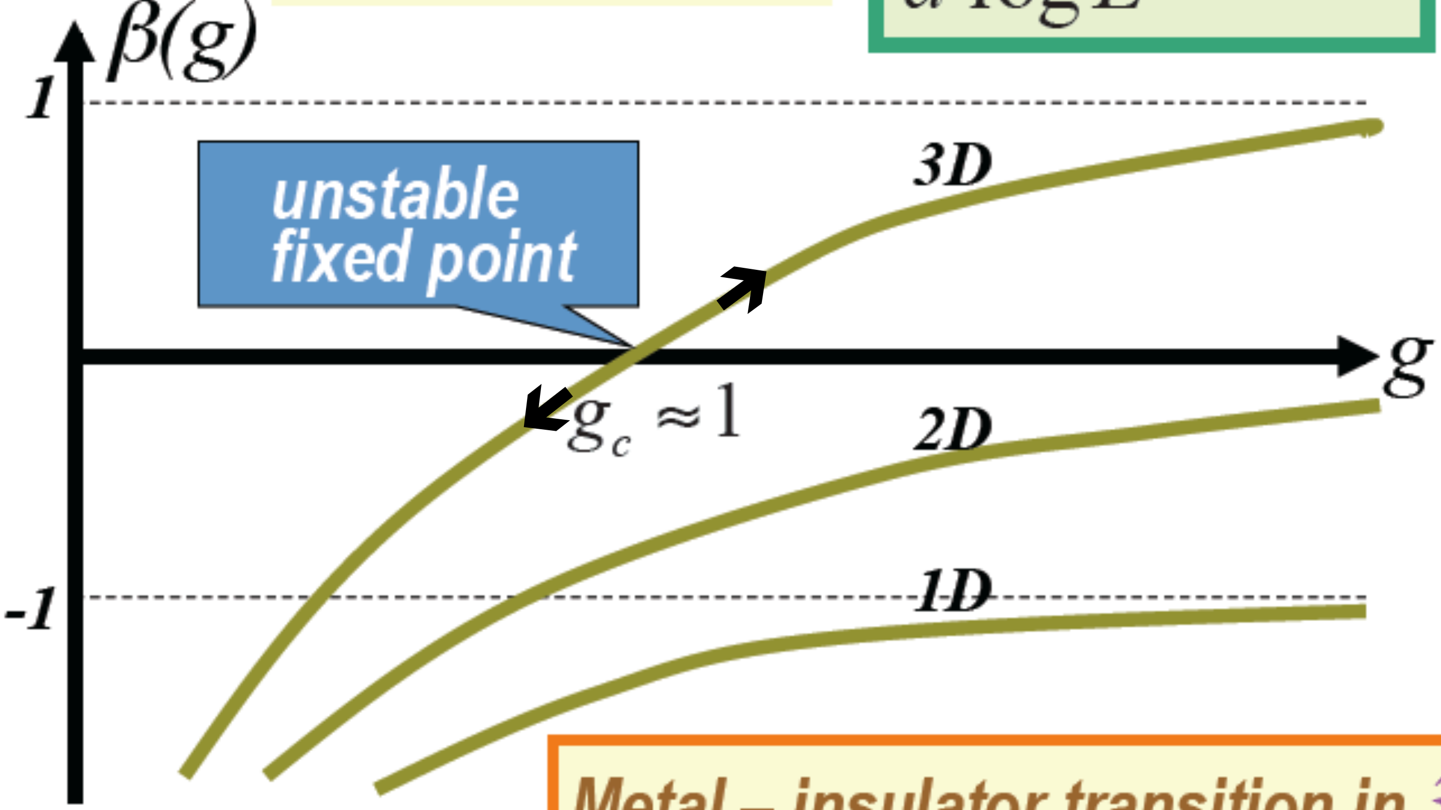
$g >> 1 \quad g \propto L^{d-2} \quad \beta(g) = (d-2) + O\left(\frac{1}{g}\right)$

$$\boxed{\begin{array}{ll} > 0 & d > 2 \\ ?? & d = 2 \\ < 0 & d < 2 \end{array}}$$

$g << 1 \quad g \propto e^{-L/\xi} \quad \beta(g) \approx \log g < 0$

# $\beta$ - function

$$\frac{d\log g}{d\log L} = \beta(g)$$



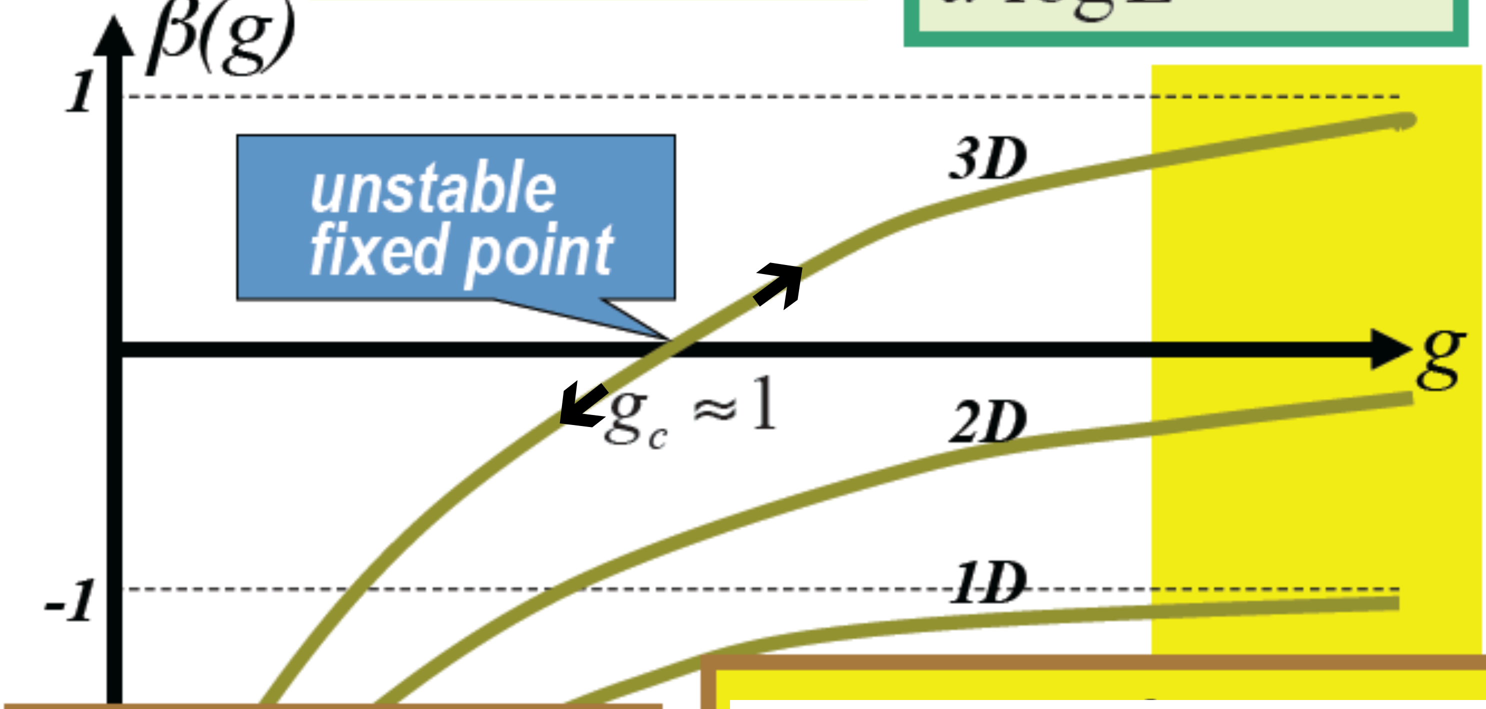
unstable fixed point

Metal – insulator transition in 3D

All states are localized for $d=1,2$

# $\beta$ - function

$$\frac{d\log g}{d\log L} = \beta(g)$$



$$\beta(g) = d - 2 + \frac{c_d}{g}$$

$c_2 = ?\quad \pm ?$

$$g(L) = g(l_s) - G_0 \ln(L/l_s)$$

$$\xi(k) = l_s e^{\pi k l_s/2}$$

Quantum corrections at large Thouless conductance – **weak localization**

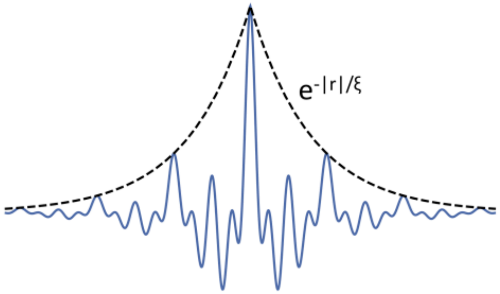
Universal description

# Localization Length

# Localization in 1 dimension:

# Transfer matrix formalism

**All eigenstates in a one-dimensional disordered lattice are localized!**



# Localization in 1 dimension

**Schrödinger equation for 1 dimensional tight-binding model:**

$$\left(\frac{-\hbar^2}{2m}\nabla^2 + V(x)\right)\psi(x) = e\psi(x)$$

$$\psi_{n+1} + \psi_{n-1} = (E - V_n)\psi_n$$

$$V_n = -\frac{2ma^2}{\hbar^2}V(na)$$

$$E = 2 - \frac{2ma^2}{\hbar^2}e$$

$$Z(n) = \begin{bmatrix} \psi_n \\ \psi_{n-1} \end{bmatrix}$$

T_n $= \begin{bmatrix} E - V_n & -1 \\ 1 & 0 \end{bmatrix}$

$$Z(n) = T_n\, T_{n-1} \cdots T_1\, Z(1)$$

**Fürstenberg's theorem for products of random matrices:**

$$\lim_{n\to\infty} \frac{1}{n} \log ||T_n \cdots T_1 \, \overrightarrow{a_0}|| = \gamma > 0$$

**Consider finite system with $N$ lattice sites:**

localization length (exponential decay length) of solution with energy E:

$$\lambda(E,N) := -\frac{1}{N} \log |a_0(E,N) a_N(E,N)|$$

**for $N \to \infty$:**

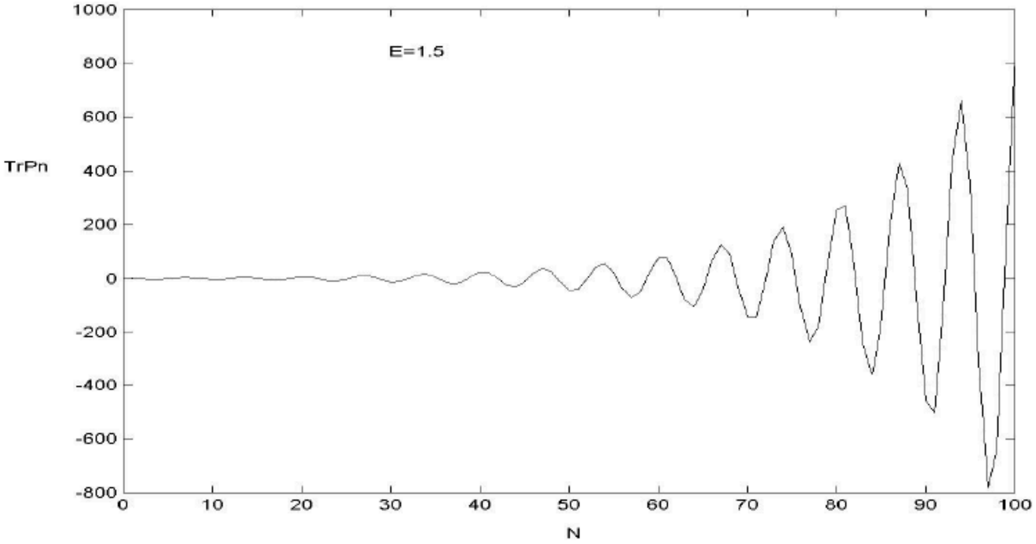
$\lambda$ can be related with $\gamma$ (no riorous proof!):

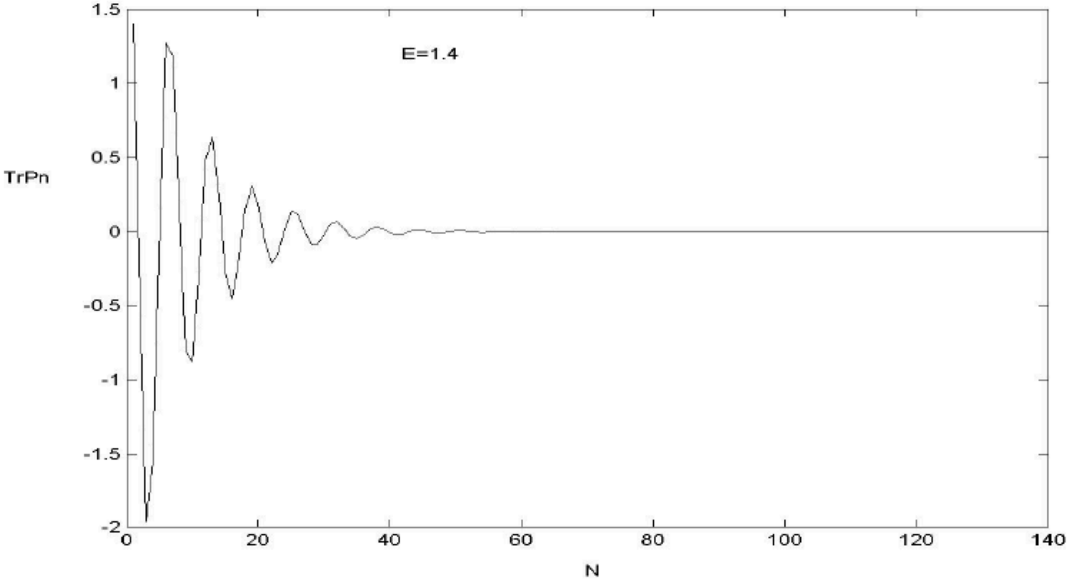
$$0 < \gamma = \lambda < \infty$$

Borland conjecture

**All eigenstates in a one-dimensional disordered lattice are localized!**

Reza Sepehrinia, et al. PRB78, 024207


E=1.5
TrPn
N
1000
800
600
400
200
0
-200
-400
-600
-800
0
10
20
30
40
50
60
70
80
90
100


E=1.4
TrPn
N
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
20
40
60
80
100
120
140

# Green's functions:

# Disordered systems of arbitrary dimension

# Green's functions

Hamiltonian $H$ with discrete eigenvalues $\{E_1, E_2, ...\}$ and eigenbasis $\{|\phi_n\rangle\}_{n\in\mathbb{N}}$
Green's Operator:

$$G(z) := \frac{1}{z-H} = \sum_n \frac{|\phi_n\rangle\langle\phi_n|}{z-E_n} \qquad \text{for } z \in \mathbb{C}\backslash\{E_1, E_2, ...\}$$

Green's function:

$$G(r, r'; z) := \langle r|G(z)|r'\rangle$$

## Green's function

contains information about system and its Hamiltonian:

- eigenvalues $E_n$ are poles of Green's function
- eigenfunctions can be deduced from the residues at the corresponding poles

# Green's functions and disordered solids

From Green's function for a disordered system we can calculate:

- Density of states $D(E)$
  (for Hamiltonian with continuous spectrum $spec(H) \subseteq \mathbb{C}$):

$$D(E) = -\frac{1}{2\pi i}\,\mathrm{Tr}[G^+(E) - G^-(E)]$$

$$G^{\pm}(E) := \lim_{\eta \to 0^+} G(E \pm i\eta)$$

- Localization length $\lambda$:

$$\frac{1}{\lambda} = -\lim_{|r-r'| \to \infty} \frac{< \log |G(r, r'; E)| >}{|r - r'|}$$

- DC electrical conductivity $\sigma$:

$$\sigma = \frac{2e^2}{h} \lim_{\eta \to 0+} 4\eta^2 \int d\mathbf{r}\, r^2 < |G^+(r; E)|^2 >$$

# Calculating Green's Functions

- Consider disorder as perturbation:

$$H = H_0 + H_1$$

$$G_0(z) = (z - H_0)^{-1}$$

$$\begin{aligned} G(z) = (z - H)^{-1} &= (z - H_0 - H_1)^{-1} = (1 - G_0(z) H_1)^{-1}\, G_0(z) \\ &= G_0 + G_0\, H_1\, G_0 + G_0\, H_1\, G_0\, H_1\, G_0 + \cdots \\ &= G_0 + G_0\, H_1\, G \\ &= G_0 + G\, H_1\, G_0 \end{aligned}$$

- Different methods to evaluate the ensemble average of the perturbation series: $< G(z) >$

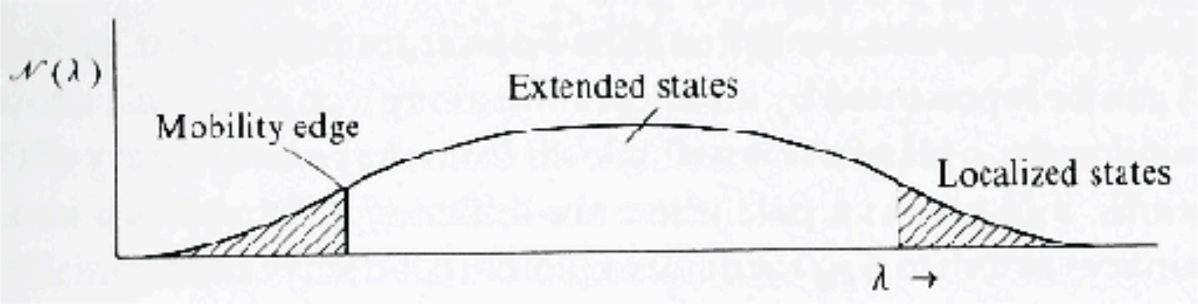
# Anderson Model: Theoretical Results

## Dimensionality of the system $d \leq 2$

All eigenstates are localized, no matter how weak the disorder!

## Dimensionality of the system $d = 3$

- DOS forms tails of the band consisting of localized states
- Interior of the band corresponds to extended states
- Critical energies $E_c$ separating localized from extended states: **Mobility edges**



**Note:** No rigorous proof of these results!

# Summary

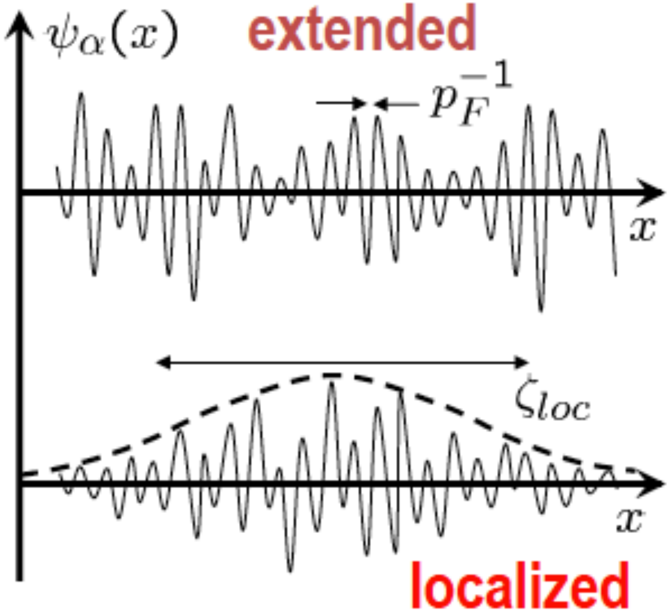
- **Perfect crystal:** Delocalized electronic eigenstates
- **1-D and 2-D disordered systems:** All eigenstates are localized!
- **3-D disordered systems:**
  Energy band forms tails of localized states: Mobility edges
- **Anderson transition:**
  Metal-Insulator phase transition at critical disorder
- **Hopping transport by localized electrons:**
  Mott's $T^{-\frac{1}{4}}$ -law for conductivity

$$\sigma(T) \propto e^{-(T_0/T)^{1/4}}$$

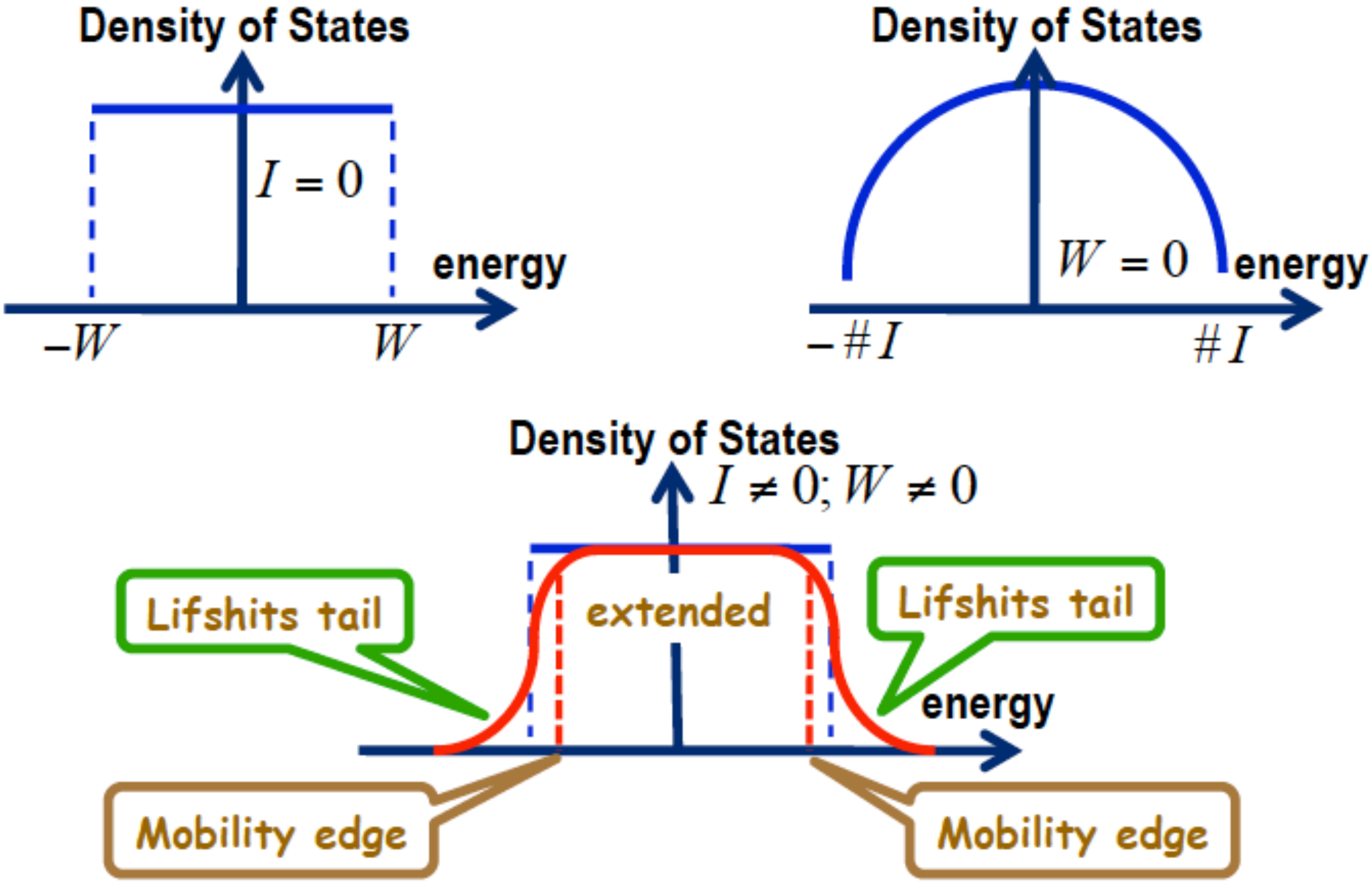
- **Weak localization:**
  Reduced conductivity in metallic regime due to disorder enhanced backscattering

# Spectral Statistics and localization

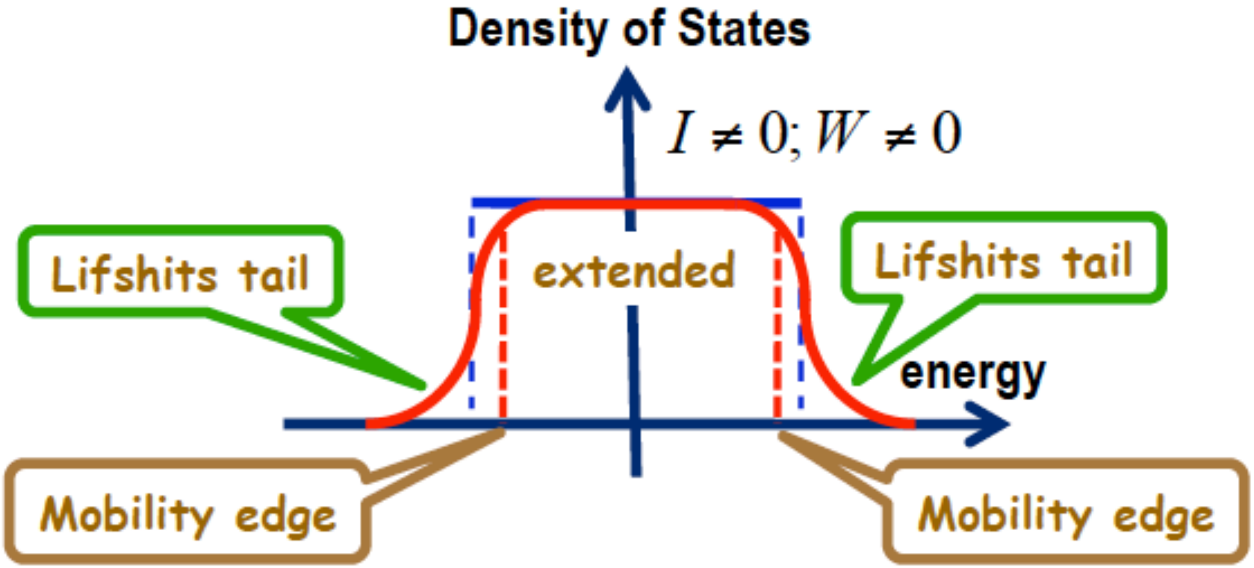
# Eigenfunctions



**Q:** Does anything interesting happen with the spectrum **?**



$$N(E) \sim \exp\left( - const.\, (E - E_0)^{-d/2} \right)$$



Density of States is not singular at the Anderson transition

# RANDOM MATRIX THEORY

**Spectral statistics**

$N \times N$ *ensemble of Hermitian matrices with random matrix element* $N \to \infty$

$E_\alpha$ - spectrum (set of eigenvalues)

$\delta_1 \equiv \langle E_{\alpha+1} - E_\alpha \rangle$ - mean level spacing

$\langle \ldots\ldots \rangle$ - ensemble averaging

$s \equiv \dfrac{E_{\alpha+1} - E_\alpha}{\delta_1}$ - spacing between nearest neighbors

$P(s)$ - distribution function of nearest neighbors spacing between

# RANDOM MATRIX THEORY

**Spectral statistics**

$N \times N$ *ensemble of Hermitian matrices with random matrix element* $N \to \infty$

$E_\alpha$ - spectrum

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$\langle \ldots\ldots \rangle$ - ensemble averaging

$s \equiv \frac{E_{\alpha+1} - E_\alpha}{\delta_1}$ - spacing between nearest neighbors

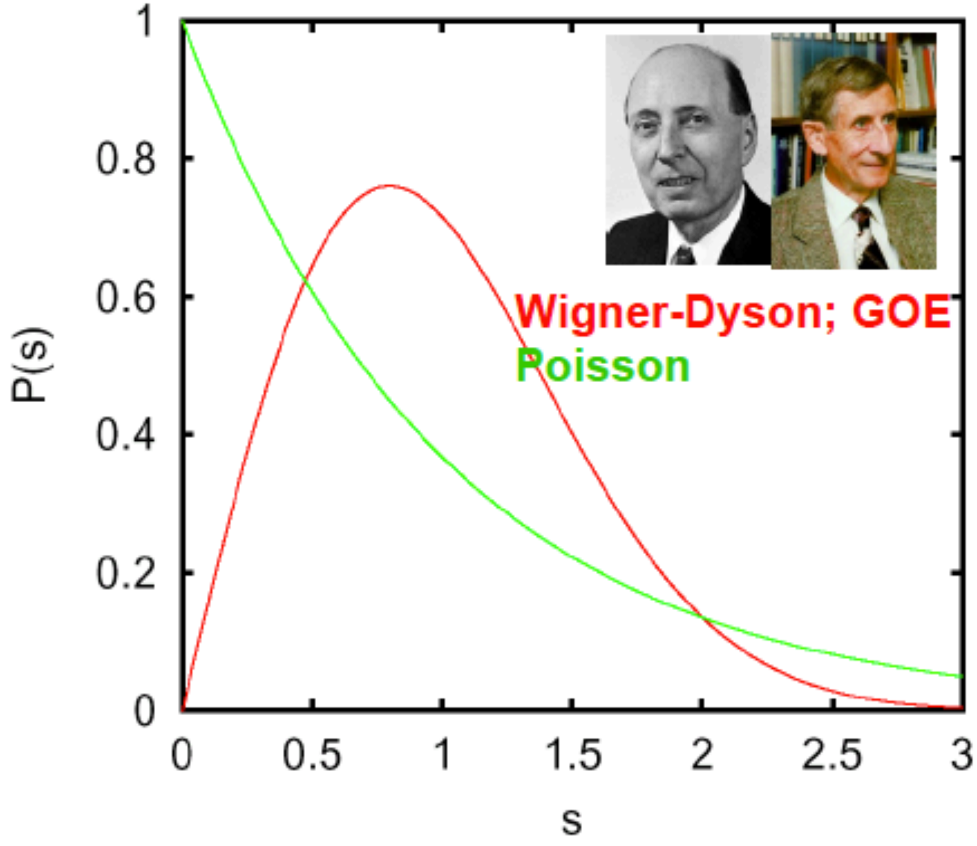
$P(s)$ - distribution function of these spacings

## Spectral Rigidity

## Level repulsion

$$P(s=0)=0$$

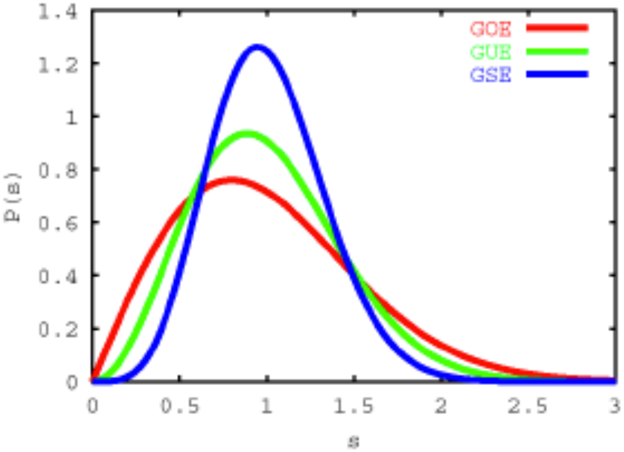
$$P(s<<1) \propto s^\beta \quad \beta=1,2,4$$



**Orthogonal**
*$\beta=1$*

**Unitary**
*$\beta=2$*

**Simplectic**
*$\beta=4$*



Poisson – random energies

# RANDOM MATRICES

$N \times N$ matrices with random matrix elements. $N \rightarrow \infty$

## Dyson Ensembles

| Matrix elements | Ensemble | $\beta$ | realization |
|---|---|---|---|
| real | orthogonal | 1 | T-inv potential |
| complex | unitary | 2 | broken T-invariance (e.g., by magnetic field) |
| $2 \times 2$ matrices | simplectic | 4 | T-inv, but with spin-orbital coupling |

localization of elastic waves, PRB 78, 024207

## Reason for $P(s) \rightarrow 0$ when $s \rightarrow 0$:

$$\hat{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^{*} & H_{22} \end{pmatrix}$$

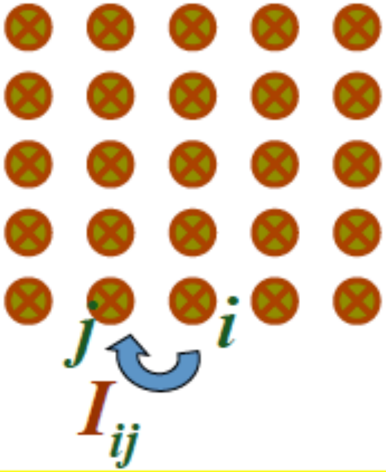
$$\underbrace{E_2 - E_1}_{\text{small}} = \sqrt{\left(\underbrace{H_{22} - H_{11}}_{\text{small}}\right)^2 + \left|\underbrace{H_{12}}_{\text{small}}\right|^2}$$

1. The assumption is that the matrix elements are statistically independent. Therefore probability of two levels to be degenerate vanishes.

2. If $H_{12}$ is real (orthogonal ensemble), then for $s$ to be small two statistically independent variables ($(H_{22}- H_{11})$ and $H_{12}$) should be small and thus $P(s) \propto s$ $\qquad \beta = 1$

3. Complex $H_{12}$ (unitary ensemble) $\Longrightarrow$ both $Re(H_{12})$ and $Im(H_{12})$ are statistically independent $\Longrightarrow$ three independent random variables should be small $\Longrightarrow$ $P(s) \propto s^2$ $\qquad \beta = 2$

# Anderson Model



- *Lattice - tight binding model*
- *Onsite energies* $\varepsilon_i$ *- random*
- *Hopping matrix elements* $I_{ij}$

$$\hat{H} = \sum_i \varepsilon_i \hat{a}_i^+ \hat{a}_i + I \sum_{i,j=n.n.} \hat{a}_i^+ \hat{a}_j$$

$$-W < \varepsilon_i < W$$

*uniformly distributed*

Q: What are the spectral statistics of a finite size Anderson model, $d>1$ ?

# Anderson Transition

**Strong disorder**

$I < I_c$

***Insulator***

***All eigenstates are localized***

***The eigenstates, which are localized at different places will not repel each other***

⇓

***Poisson spectral statistics***

**Weak disorder**

$I > I_c$

***Metal***

***Extended states***

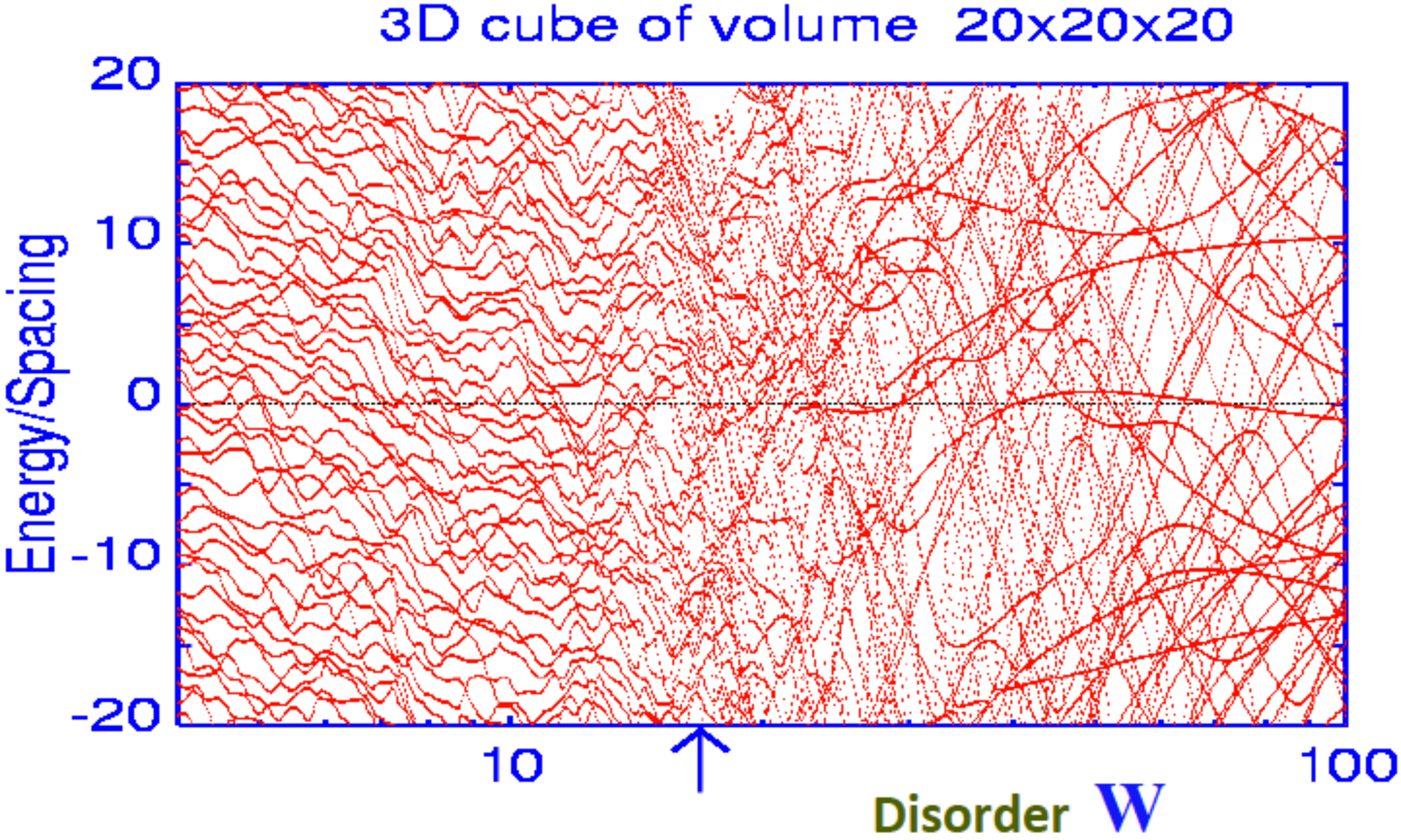
***Any two extended eigenstates repel each other***

⇓

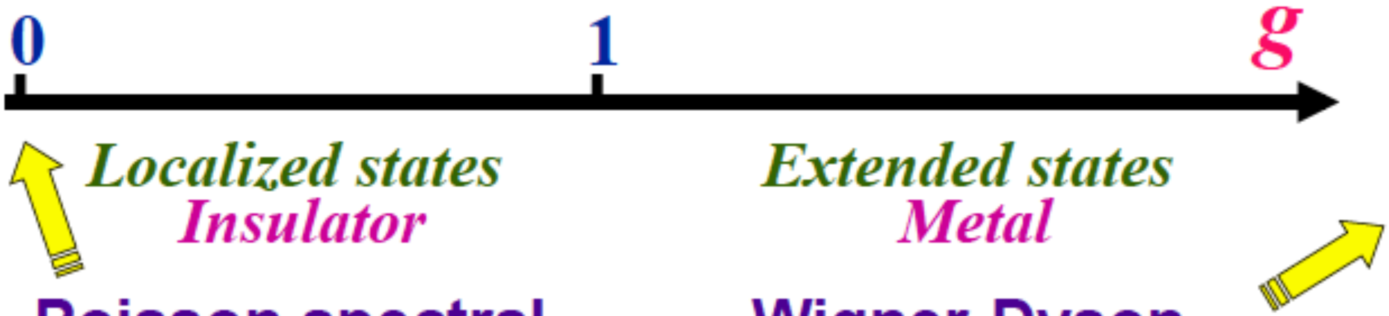
***Wigner – Dyson spectral statistics***

Zharekeschev & Kramer.

*Exact diagonalization of the Anderson model*



# Thouless Conductance and One-particle Spectral Statistics



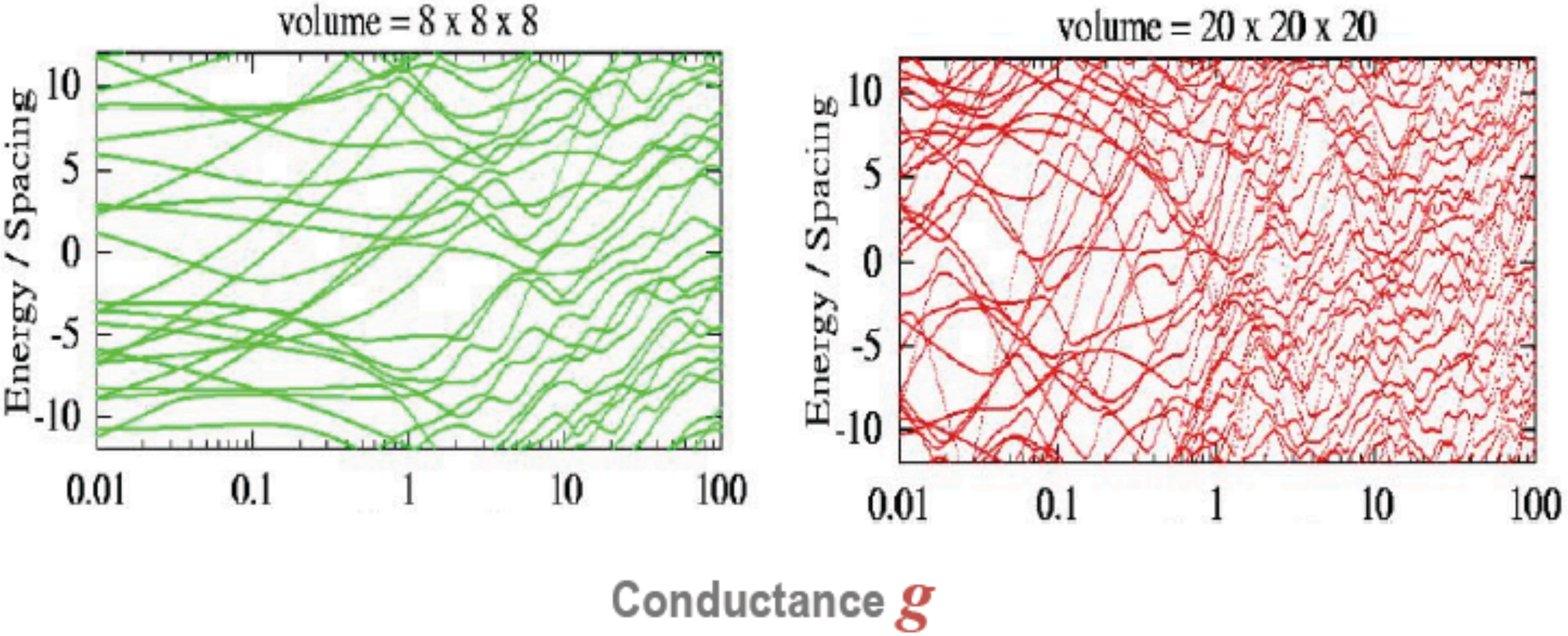
*Localized states*
*Insulator*

**Poisson spectral statistics**

*Extended states*
*Metal*

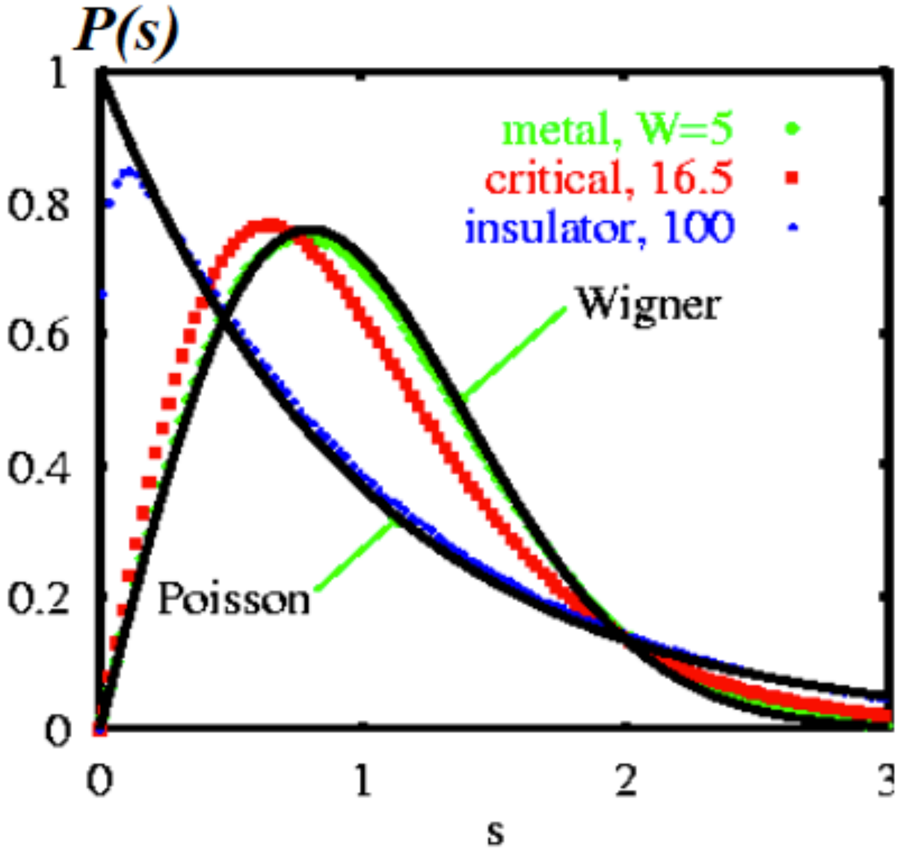
**Wigner-Dyson spectral statistics**

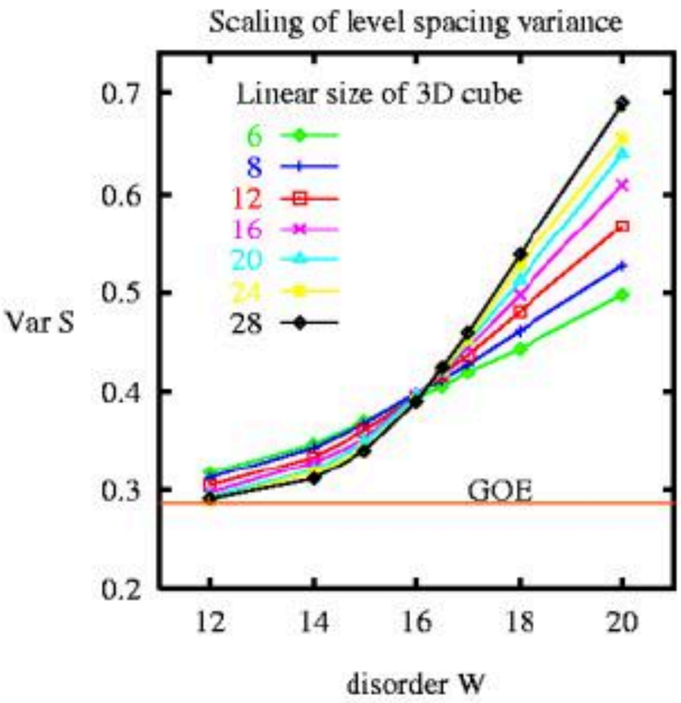
Transition at $g \sim 1$. Is it sharp?



**The bigger the system the sharper the transition**

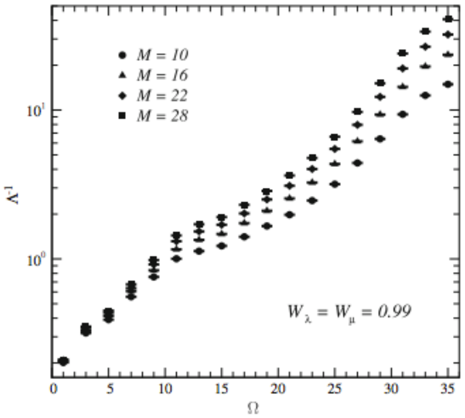
# Anderson transition in terms of pure level statistics




Scaling of level spacing variance

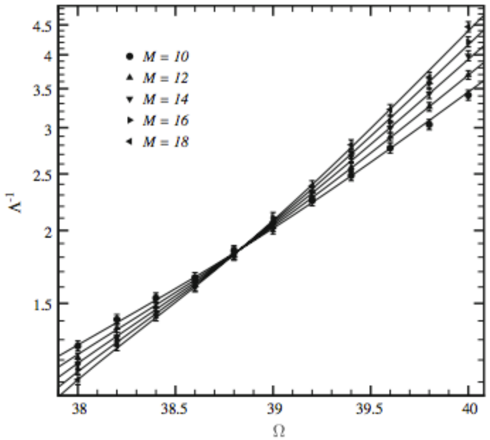
$$\gamma \sim |W - W_c|^{\nu}$$

2d



$M \times L$ strips

3d



$M \times M \times L$ bars,

Sahimi et al. Acta Mech 205, 197 (2009)

# Distribution of the local density of states as a criterion for Anderson localization:

Schubert, et al. PRB81, 155106 (2010)

# Distribution of the local density of states as a criterion for Anderson localization:

- Numerical approaches to Anderson localization face the problem of having to treat large localization lengths while being restricted to finite system sizes.

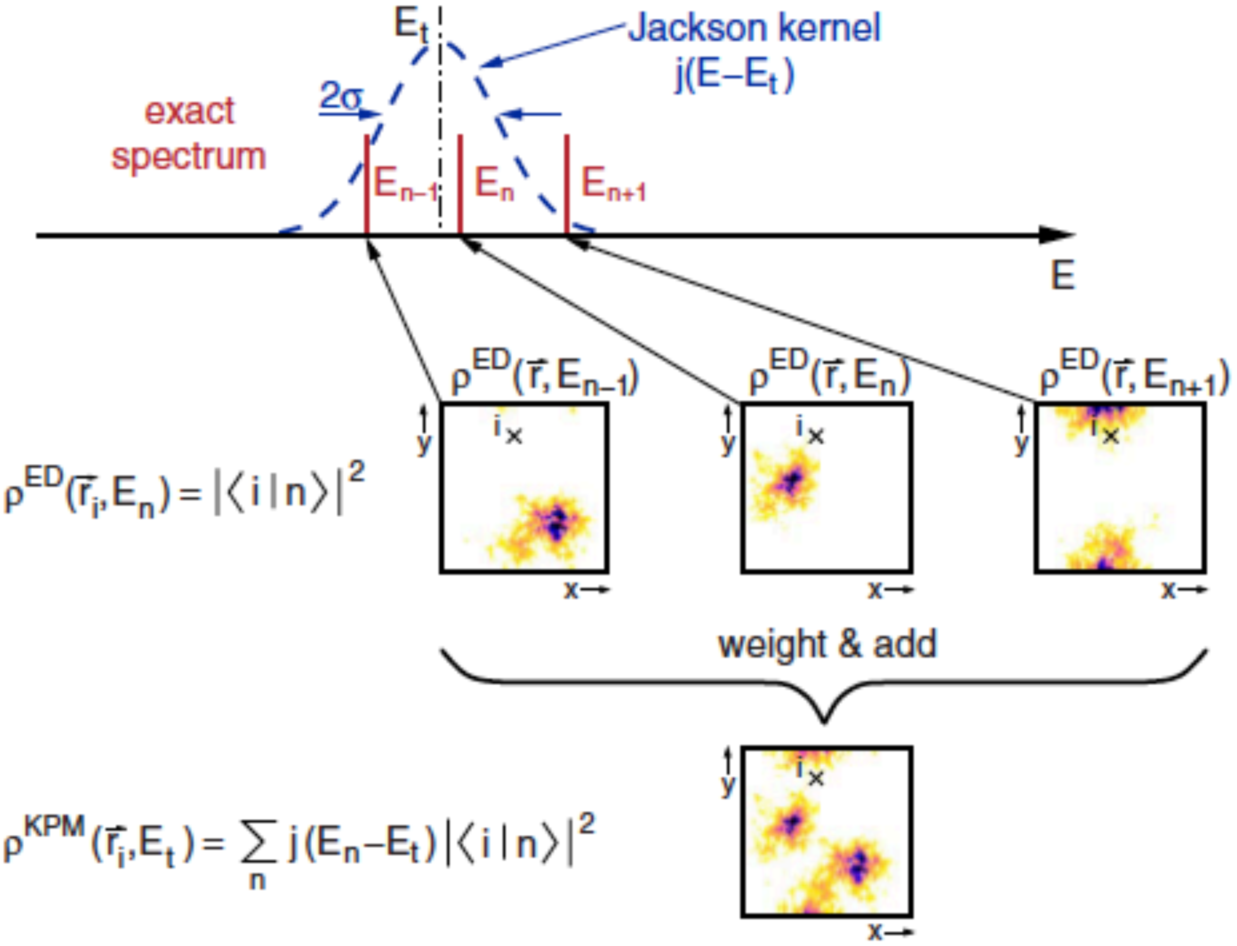
- It is shown that the system-size dependence of the LDOS distribution is sign of Anderson localization, irrespective of the dimension and lattice structure.

- The numerically obtained exact LDOS data is agree with a log-normal distribution over up to ten orders of magnitude

$$H = -t\sum_{\langle ii\rangle} (c_i^\dagger c_j + \text{H.c.}) + \sum_i \epsilon_i c_i^\dagger c_i,$$

$$\rho_i(E) = \sum_{n=1}^{N} |\langle i|n\rangle|^2 \delta(E - E_n),$$

$$p[\epsilon_i] = \frac{1}{\gamma}\theta\left(\frac{\gamma}{2} - |\epsilon_i|\right).$$



Exact diagonalization ED

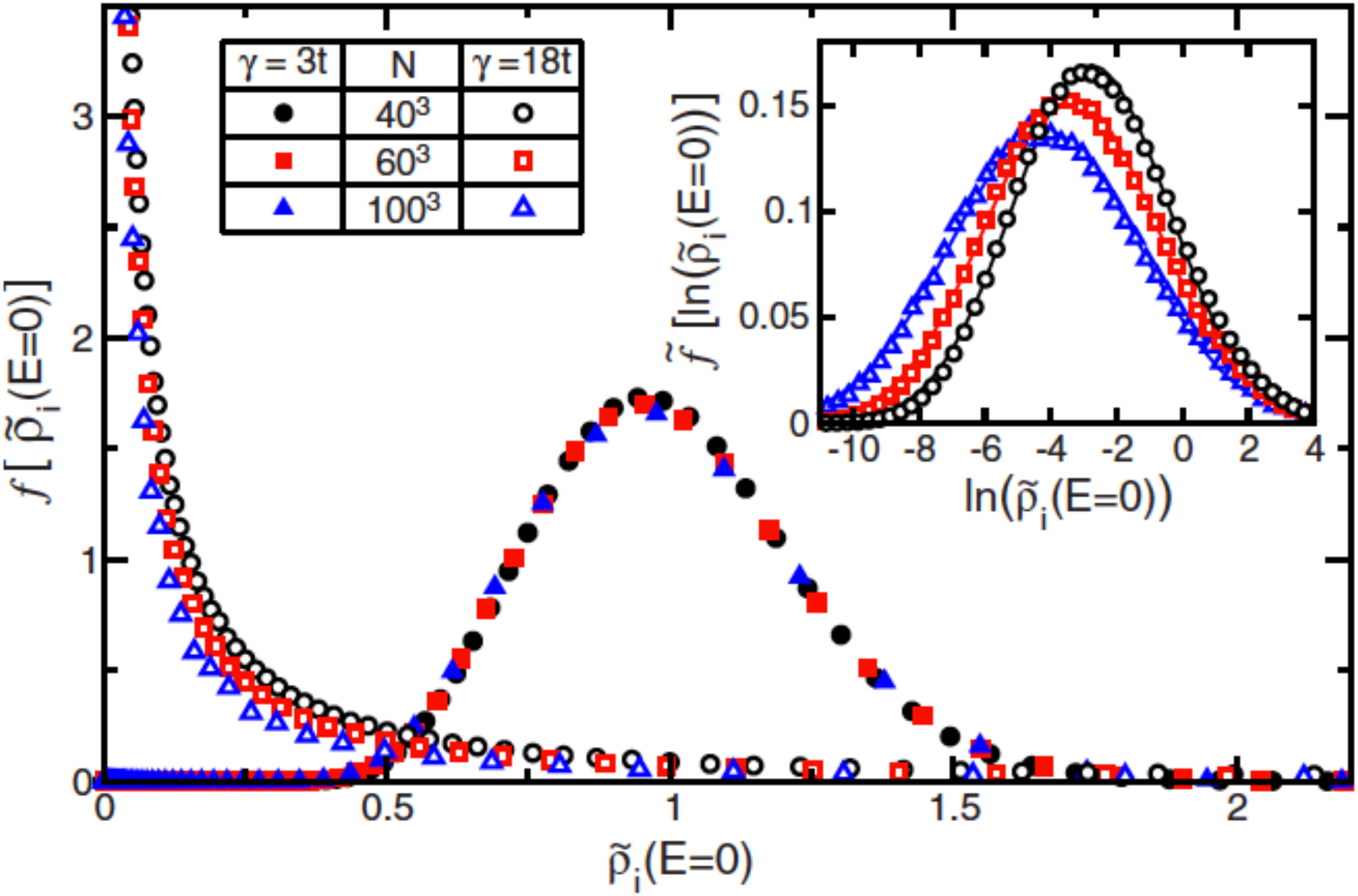
Kernel Polynomial Method KPM

$$\rho_i(E) = \frac{1}{D} \sum_{k=0}^{D-1} |\langle i|k\rangle|^2 \, \delta(E - E_k) \, ,$$

$$\mu_n = \int_{-1}^{1} \tilde{\rho}_i(E) \, T_n(E) \, dE = \frac{1}{D} \sum_{k=0}^{D-1} |\langle i|k\rangle|^2 T_n(\tilde{E}_k)$$

$$= \frac{1}{D} \sum_{k=0}^{D-1} \langle i|T_n(\tilde{M})|k\rangle\langle k|i\rangle = \frac{1}{D} \, \langle i|T_n(\tilde{M})|i\rangle \, .$$

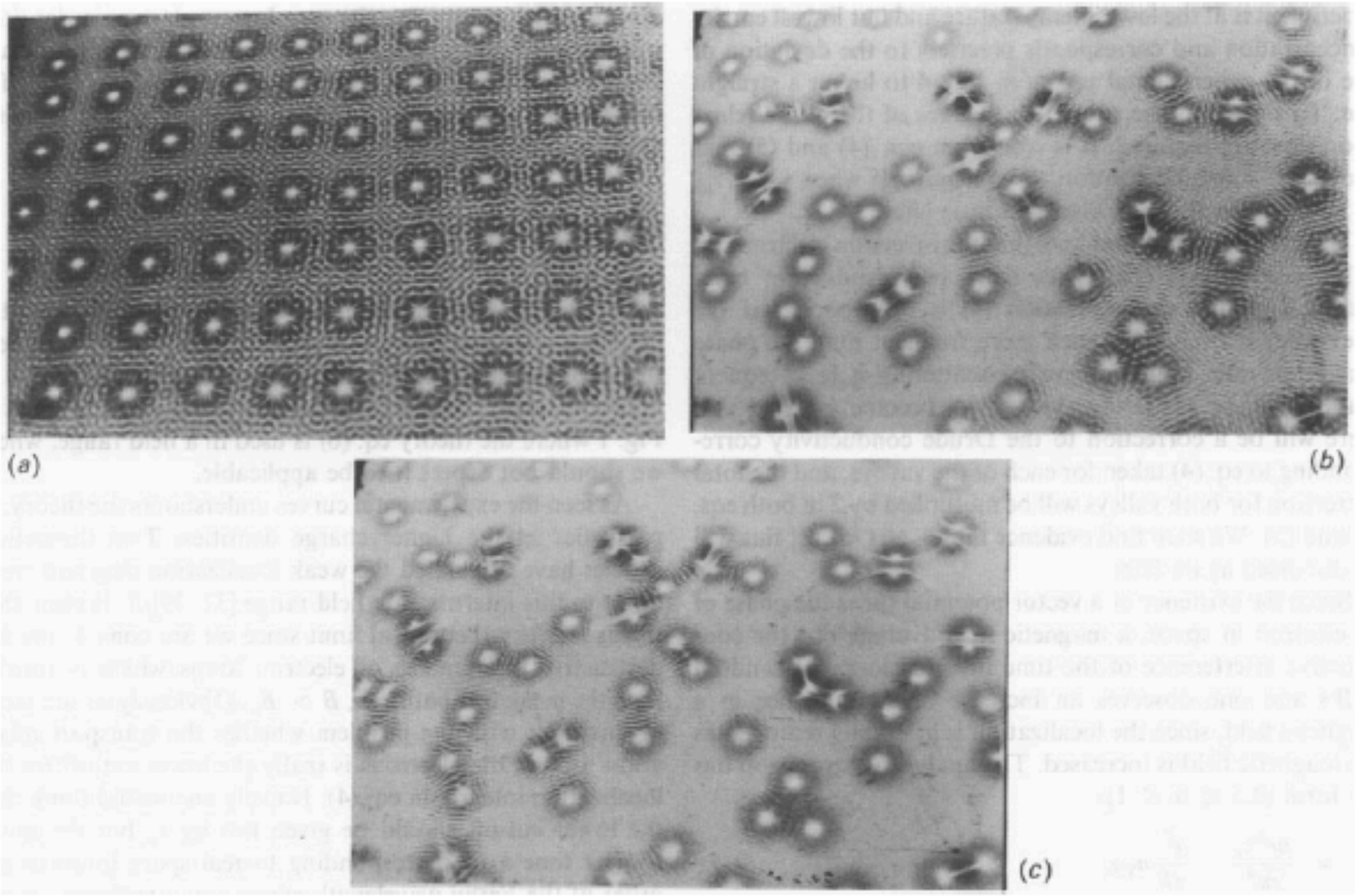
$$T_n(x) = \cos(n \arccos(x)) \, ,$$


γ = 3t
N
γ =18t
40^3
60^3
100^3
f [ ρ̃_i(E=0)]
ρ̃_i(E=0)
0
0.5
1
1.5
2
0
1
2
3
f̃ [ln(ρ̃_i(E=0))]
ln(ρ̃_i(E=0))
0
0.05
0.1
0.15
-10
-8
-6
-4
-2
0
2
4

# Classical Wave Localization

**Problem:** Electrons interact
With each other

# Randomly scattered water waves



- regularly distributed scatterers: waves are spreading all over the surface
- randomly distributed scatterers: waves remain localized in some areas

# Experiment

## Spin Diffusion

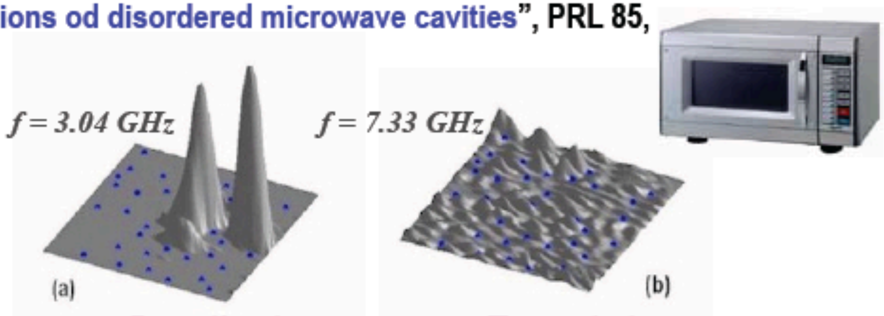
Feher, G., Phys. Rev. 114, 1219 (1959); Feher, G. & Gere, E. A., Phys. Rev. 114, 1245 (1959).

## Microwave

Dalichaouch, R., Armstrong, J.P., Schultz, S.,Platzman, P.M. & McCall, S.L. "Microwave localization by 2-dimensional random scattering". *Nature* 354, 53-55, (1991).

Chabanov, A.A., Stoytchev, M. & Genack, A.Z. Statistical signatures of photon localization. *Nature* 404, 850-853 (2000).

Pradhan, P., Sridar, S, "Correlations due to localization in quantum eigenfunctions od disordered microwave cavities", PRL 85,



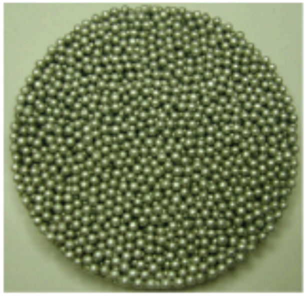
*Localized* *Extended*

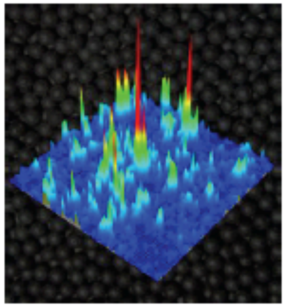
## Experiment

# Localization of Ultrasound

**Weaver, R.L. "Anderson localization of ultrasound".**
Wave Motion 12, 129-142 (1990).

**H. Hu, A. Strybulevych, J. H. Page, S. E. Skipetrov & B. A. van Tiggelen** "**Localization of ultrasound in a three-dimensional elastic network**" Nature Phys. 4, 945 (2008).



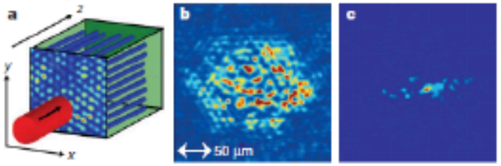


# Localization of Light

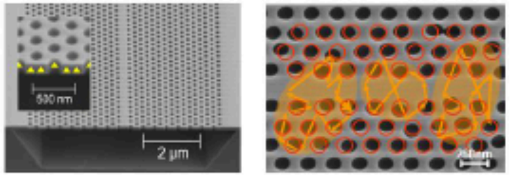
**D. Wiersma, Bartolini, P., Lagendijk, A. & Righini R. "Localization of light in a disordered medium",** Nature 390, 671-673 (1997).

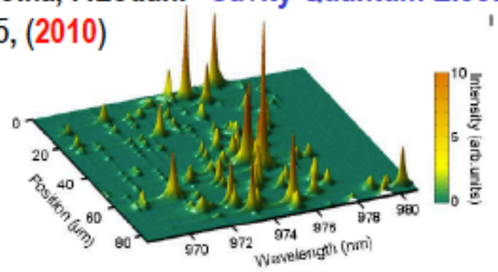
**Scheffold, F., Lenke, R., Tweer, R. & Maret, G. "Localization or classical diffusion of light",** Nature 398,206-270 (1999).

**Schwartz, T., Bartal, G., Fishman, S. & Segev, M. "Transport and Anderson localization in disordered two dimensional photonic lattices".** Nature 446, 52-55 (2007).



**L.Sapienza, H.Thyrrestrup, S.Stobbe, P. D.Garcia, S.Smolka, P.Lodahl "Cavity Quantum Electrodynamics with Anderson localized Modes"** Science 327, 1352-1355, (2010)

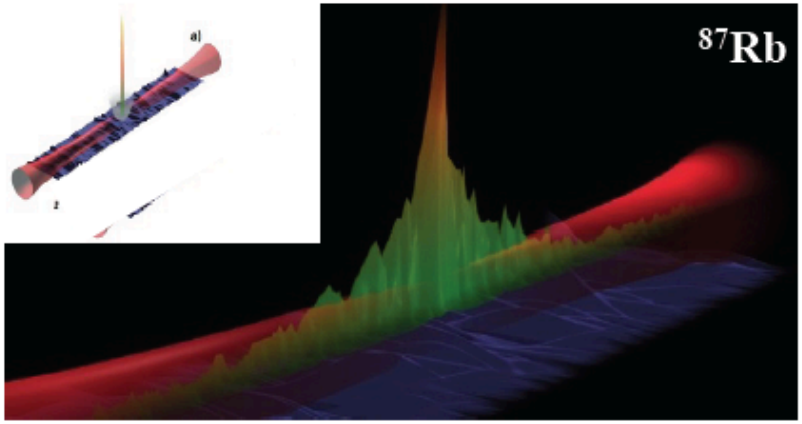




# Note: all of the previous examples are classical waves

## Localization of cold atoms

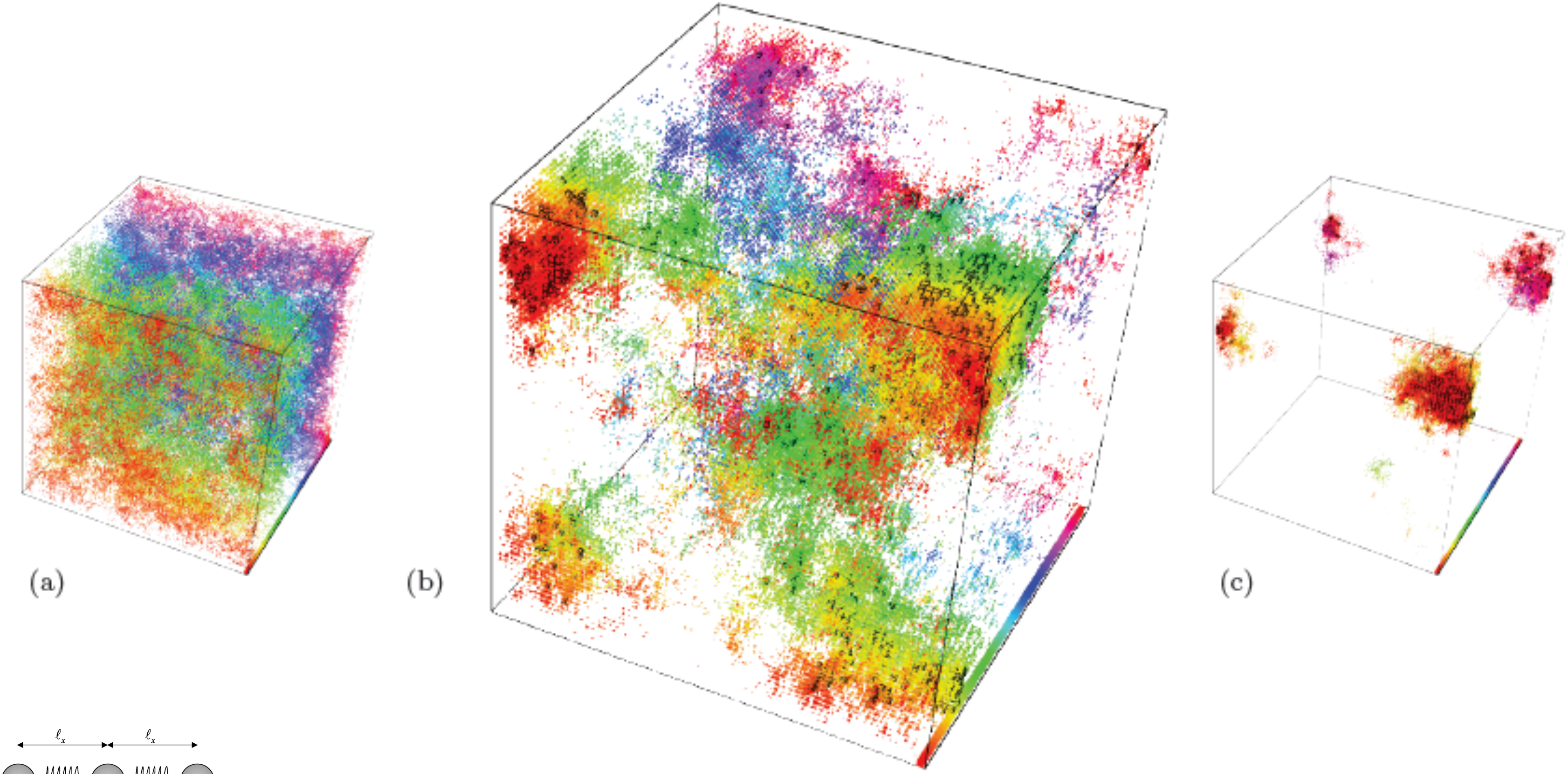
Billy et al. “Direct observation of Anderson localization of matter waves in a controlled disorder”. Nature 453, 891- 894 (2008).

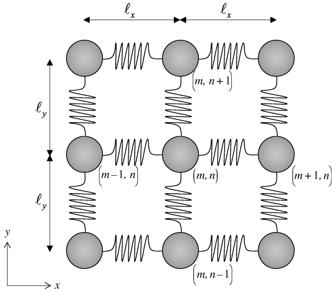


Roati et al. “Anderson localization of a non-interacting Bose-Einstein condensate“. Nature 453, 895-898 (2008).

# Random mass

$$E \leftrightarrow 6 - \omega^2, \qquad \epsilon_j(E) \leftrightarrow \omega^2 m_j = (6 - E)m_j\,.$$





PRB78, 134206 (2008)

# Random spring constants

$$\frac{\partial^2}{\partial t^2}\psi(x,t) - \frac{\partial}{\partial x}\left[\lambda(x)\frac{\partial}{\partial x}\psi(x,t)\right] = 0,$$

$$\lambda_{i+1/2}(\psi_{i+1} - \psi_i) - \lambda_{i-1/2}(\psi_i - \psi_{i-1}) + \omega^2\psi_i = 0.$$

$$\lambda_{i+1/2} = \beta_i,$$

$$\lambda_{i-1/2} = \beta_{i-1},$$

$$\begin{pmatrix} \psi_{i+1} \\ \psi_i \end{pmatrix} = \mathbf{M}_{i,i-1}\begin{pmatrix} \psi_i \\ \psi_{i-1} \end{pmatrix},$$

$$\mathbf{M}_{i,i-1} = \begin{pmatrix} \dfrac{-\omega^2 + \beta_{i-1} + \beta_i}{\beta_i} & -\dfrac{\beta_{i-1}}{\beta_i} \\ 1 & 0 \end{pmatrix}.$$

PRL 94, 165505 (2005)

# Elastic wave localization

# Martin-Siggia-Rose action

PRL 94, 165505 (2005) PHYSICAL REVIEW LETTERS week ending 29 APRIL 2005

# Localization of Elastic Waves in Heterogeneous Media with Off-Diagonal Disorder and Long-Range Correlations

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Using the Martin-Siggia-Rose method, we study propagation of acoustic waves in strongly heterogeneous media which are characterized by a broad distribution of the elastic constants. Gaussian-white distributed elastic constants, as well as those with long-range correlations with nondecaying power-law correlation functions, are considered. The study is motivated in part by a recent discovery that the elastic moduli of rock at large length scales may be characterized by long-range power-law correlation functions. Depending on the disorder, the renormalization group (RG) flows exhibit a transition to localized regime in *any* dimension. We have numerically checked the RG results using the transfer-matrix method and direct numerical simulations for one- and two-dimensional systems, respectively.

# The Model (Scalar Field)

The scalar wave equation:

$$\frac{\partial^2}{\partial t^2}\psi(\mathbf{x},t) - \boldsymbol{\nabla}\cdot[\lambda(\mathbf{x})\boldsymbol{\nabla}\psi(\mathbf{x},t)] = 0 \ , \tag{2}$$

where $\psi(\mathbf{x},t)$ is the wave amplitude, and $\lambda(\mathbf{x}) = e(\mathbf{x})/m$ the ratio of the elastic stiffness $e(\mathbf{x})$ and the medium's mean density $m$. We then write $\lambda$ as,

$$\lambda(\mathbf{x}) = \lambda_0 + \eta(\mathbf{x}) \ , \tag{3}$$

where $\lambda_0 = \langle\lambda(\mathbf{x})\rangle$. In the present paper $\eta(\mathbf{x})$ is assumed to be a Gaussian random process with a zero mean and the covariance,

$$\langle\eta(\mathbf{x})\eta(\mathbf{x}')\rangle = 2C(|\mathbf{x}-\mathbf{x}'|) = 2D_0\delta^d(\mathbf{x}-\mathbf{x}') + 2D_\rho|\mathbf{x}-\mathbf{x}'|^{2\rho-d}.$$

# *Propagation of Wave Component with Frequency ω*

$$\nabla^2\psi(\mathbf{x},\omega) + \frac{w^2}{\lambda_0}\psi(\mathbf{x},\omega) + \nabla\cdot(\frac{\eta(\mathbf{x})}{\lambda_0}\nabla\psi(\mathbf{x},\omega)) = 0$$

# *The Martin-Siggia-Rose Action*

$$S_e[\psi_I, \psi_R, \tilde{\psi}, \chi, \chi^*] =$$

$$\int d\mathbf{x}d\mathbf{x}' \left[ (i\tilde{\psi}_I(\mathbf{x}')(\nabla^2 + \frac{\omega^2}{\lambda_0})\psi_I(\mathbf{x}) + i\tilde{\psi}_R(\mathbf{x}')(\nabla^2 + \frac{\omega^2}{\lambda_0})\psi_R(\mathbf{x}) \right.$$

$$+ \chi^*(\mathbf{x}')(\nabla^2 + \frac{\omega^2}{\lambda_0})\chi(\mathbf{x}))\delta(\mathbf{x} - \mathbf{x}')$$

$$+ (i\boldsymbol{\nabla}\tilde{\psi}_I\boldsymbol{\nabla}\psi_I + i\boldsymbol{\nabla}\tilde{\psi}_R\boldsymbol{\nabla}\psi_R + \boldsymbol{\nabla}\chi\boldsymbol{\nabla}\chi)\frac{K(\mathbf{x} - \mathbf{x}')}{\lambda_0^2}\left(i\boldsymbol{\nabla}\tilde{\psi}_I\boldsymbol{\nabla}\psi_I\right.$$

$$\left.\left. + i\boldsymbol{\nabla}\tilde{\psi}_R\boldsymbol{\nabla}\psi_R + \boldsymbol{\nabla}\chi\boldsymbol{\nabla}\chi\right)\right]$$

Two coupling constants:

$g_0 = D_0/\lambda_0^2$, $g_\rho = D\rho/\lambda_\rho^2$

RG analysis to one-loop order in the limit, $\omega^2/\lambda_0 \rightarrow 0$, to determine the two beta functions.

# *Diagrammatic Representation and One-Loop Corrections*

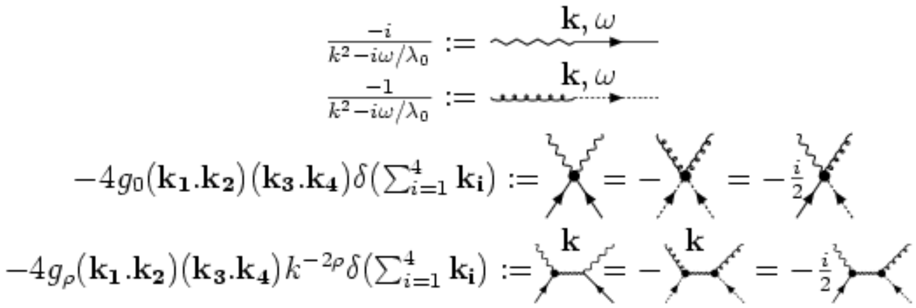


FIG. 2. Diagramatic representations of the propagators and vortices in the effective action $S_e$.

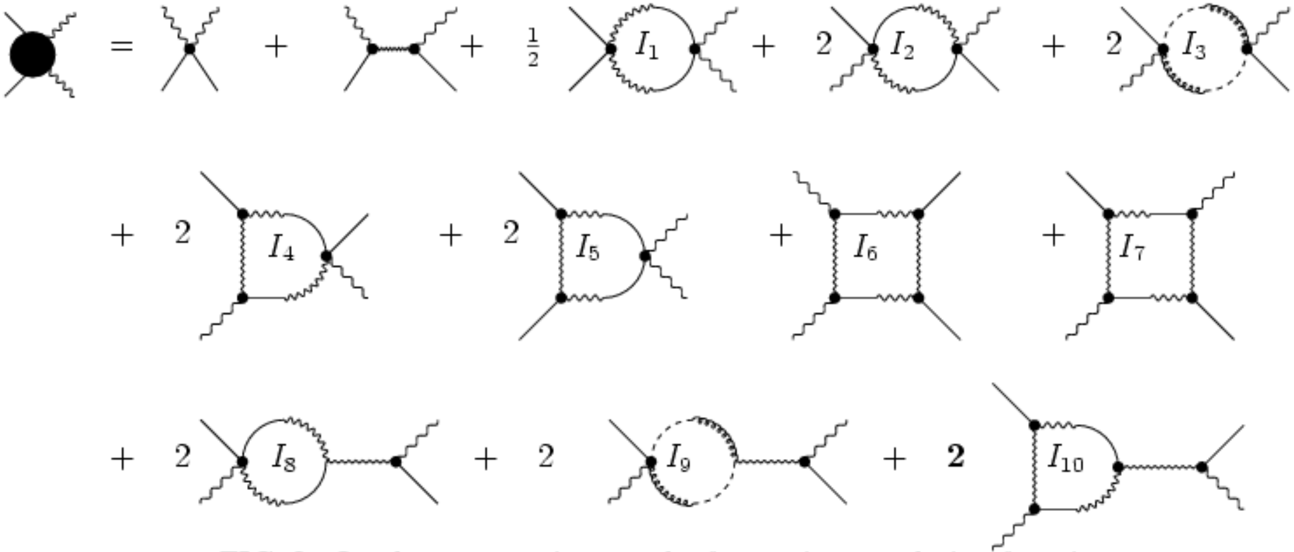


FIG. 3. One-loop corrections to the four-point correlation function.

# *The Beta Functions*

The functions $\beta(\tilde{g}_0)$ and $\beta(\tilde{g}_\rho)$ are then given by,

$$\beta(\tilde{g}_0) = \frac{\partial \tilde{g}_0}{\partial \ln l} = -d\tilde{g}_0 + 8\tilde{g}_0^2 + 10\tilde{g}_\rho^2 + 20\tilde{g}_0\tilde{g}_\rho \; ,$$

$$\beta(\tilde{g}_\rho) = \frac{\partial \tilde{g}_\rho}{\partial \ln l} = (2\rho - d)\tilde{g}_\rho + 12\tilde{g}_0\tilde{g}_\rho + 16\tilde{g}_\rho^2 \; ,$$

where $l > 1$ is the re-scaling parameter, and

$$\tilde{g}_0 = k_d \left[ \frac{d+5}{2d(d+2)} \right] g_0 \; ,$$

$$\tilde{g}_\rho = k_d \left[ \frac{d+5}{2d(d+2)} \right] g_\rho \; ,$$

# *Phase Space and Fixed Points*

Three sets of fixed points for $0 < \rho < d/2$:

Trivial FP (Gaussian) at $g_0^* = g_\rho^* = 0$ (stable)

Non-trivial FPs, one at $g_0^* = d/8$, $g_\rho^* = 0$, and the other at

$$\begin{aligned} g_0^* =& -\frac{4}{41}\left[d + \frac{5}{16}(2\rho - d)\right] \\ & -\frac{4}{41}\sqrt{\left[d + \frac{5}{16}(2\rho - d)\right]^2 + \frac{205}{256}(2\rho - d)^2}\,, \\ g_\rho^* =& \frac{3}{4}g_0^* + \frac{1}{16}(d - 2\rho)\,, \end{aligned}$$

Stable in one eigendirection, but unstable in the other eigendirection.

# *Phase Space (Continued)*

Thus, a system with uncorrelated disorder is unstable against disorder with long-range correlations towards a new FP.

Thus, with increasing disorder, extended → localized

# *Phase Space (Continued)*

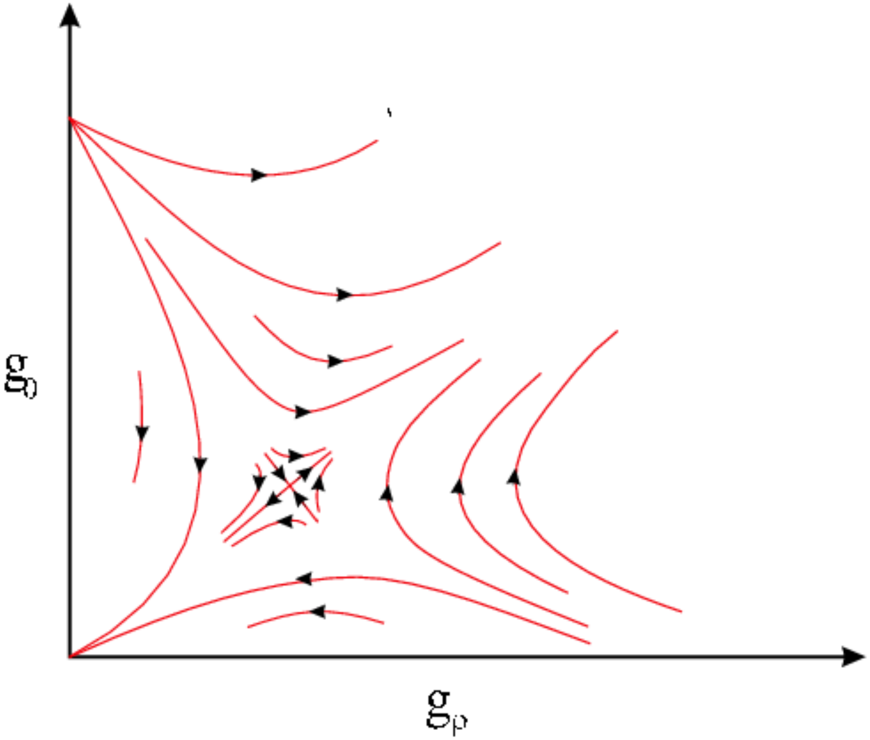


FIG. 5. Flows in the coupling constants space for $0 < \rho < \frac{1}{2}d$.

# *Phase Space (Continued)*

Two sets of fixed points for $\rho > d/2$:

Gaussian FP, stable on the $g_0$ axis, but not on the $g_\rho$ axis

Non-trivial FP at $g_0^* = d/8$, $g_\rho^* = 0$, unstable in all directions.

Thus, power-law disorder relevant, but no new FP.

Long-wavelength behavior determined by long-range correlations.

# *Phase Space (Continued)*

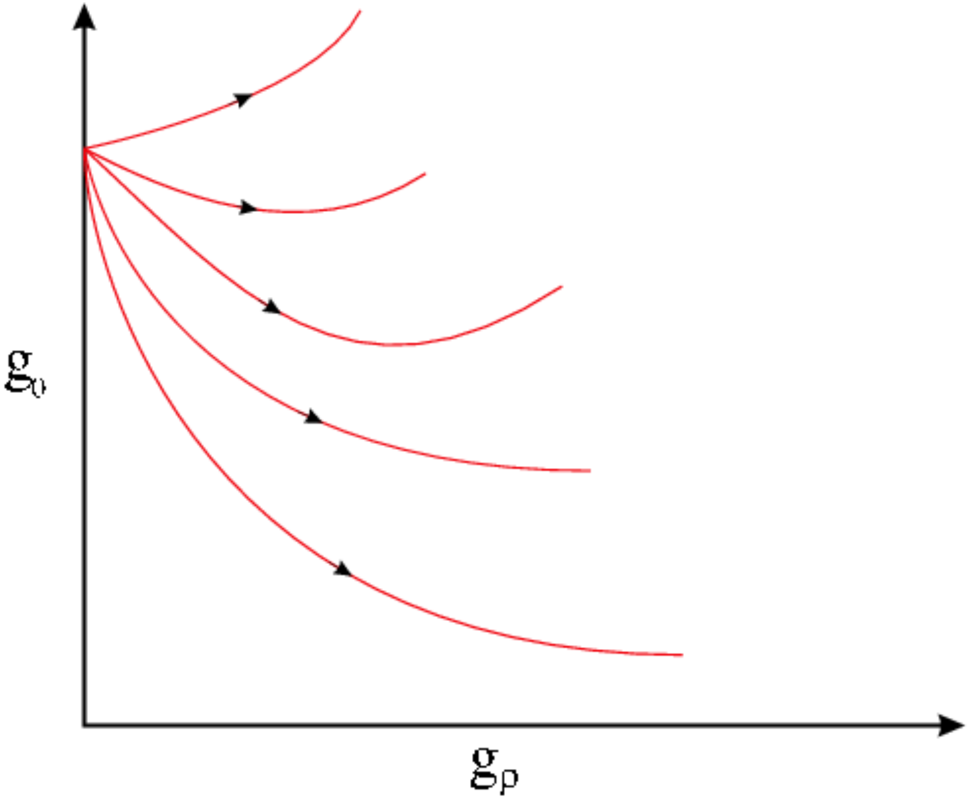
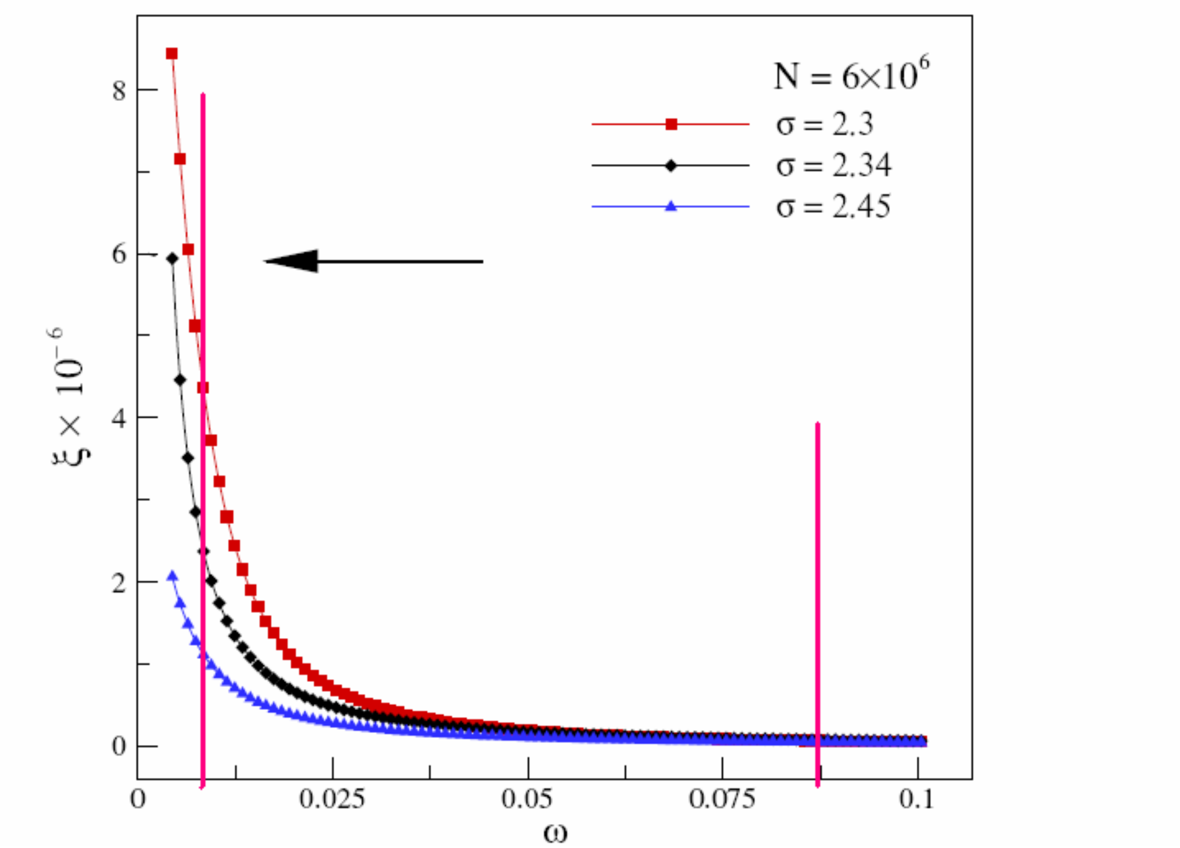


FIG. 6. Flows in the coupling constants space for $\rho > \frac{1}{2}d$.

# *Frequency-Dependence of Localization Length*



Localization length $\xi$ as a function of the frequency $\omega$ for $\sigma < \sigma_c \simeq 2.4$ and $\sigma > \sigma_c$.

The system size is $N = 6 \times 10^6$. The results represent averages over 6000 realizations.

# *Wave Front*

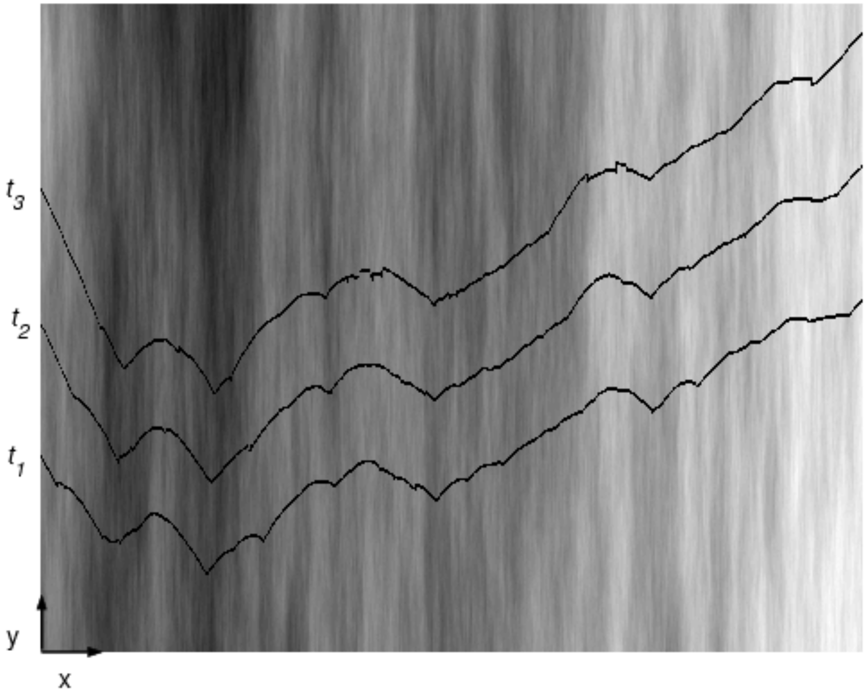


FIG. 7. The wave front in a 2D anisotropic system at (dimensionless) times, $t_1 = 328$, $t_2 = 384$, and $t_3 = 440$, with $\rho = 1.5$.

# *Roughness of Wave Front: Self-Affine Fronts*

Computing the correlation function

$$C(r) = < [d(x) - d(x + r)]^2 >$$

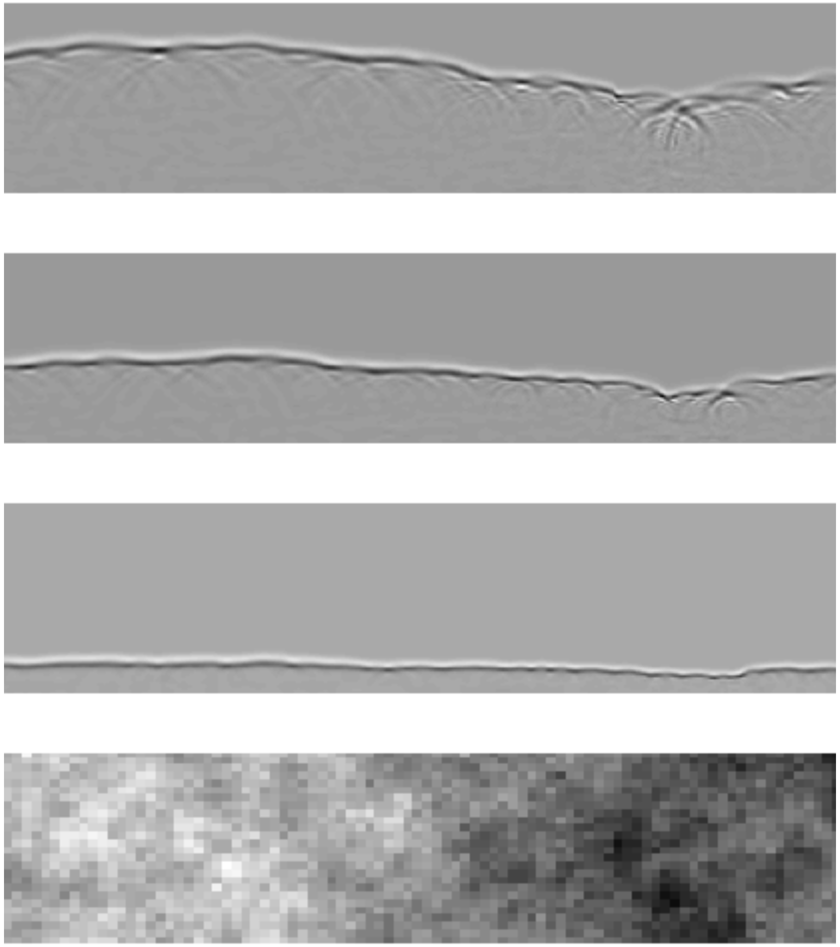
d(x) = distance from the source along the propagation direction

$$C(r) \sim r^{2\alpha}$$

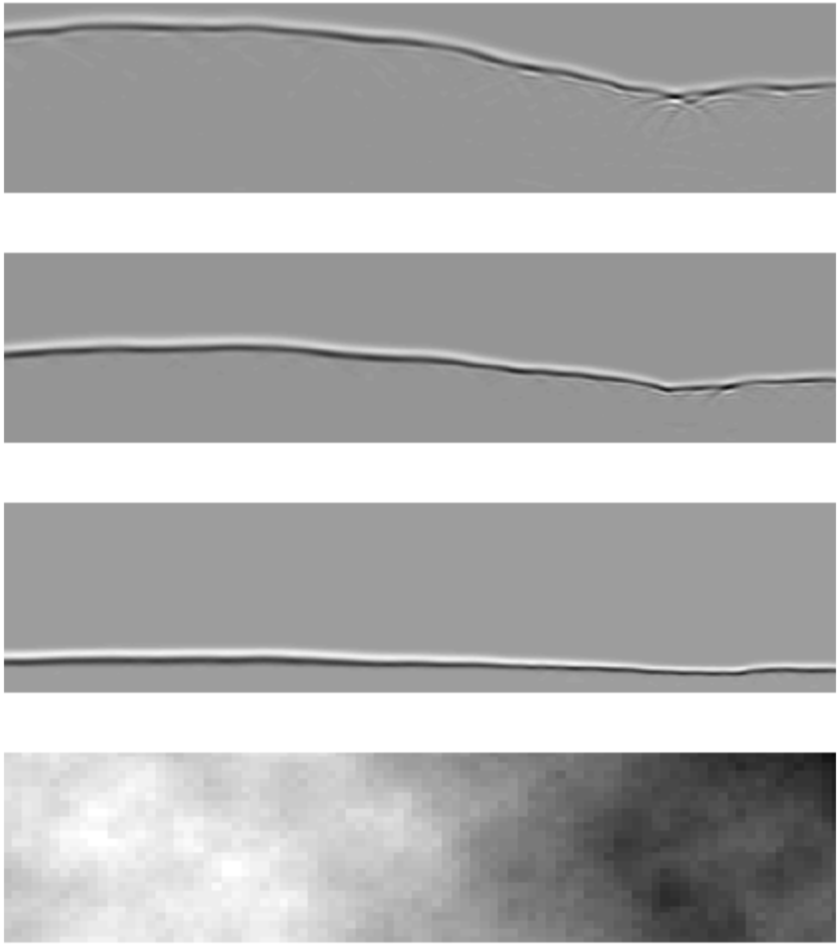
$$\alpha = H = \rho\text{-}1$$

S. M. Vaez Allaei and M. Sahimi, PRL 96, 075507 (2006).

# The Shape of Wave Front and its Evolution



*H*=0.3

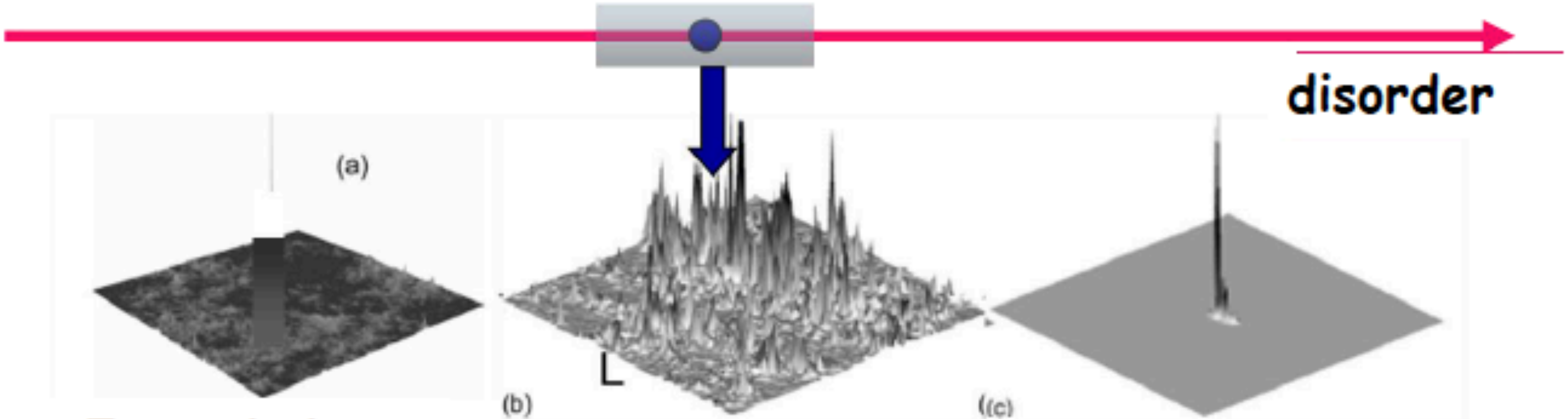


*H*=0.75

# Multifractality

# 3D Anderson transition

Anderson transition



disorder

Extended
ERGODIC states

Critical states

Localized
states

Multifractal Critical states -only at the transition point

## Localized state wavefunctions have a complex spatial structure and exhibit multifractility

$$P_q = \int_L d^d r |\psi(\mathrm{r})|^{2\mathrm{q}}$$

Inverse participation ratio

$$\langle P_q \rangle \sim \begin{cases} L^0 & \text{Insulator} \\ L^{-\tau_q} & \text{Critical} \\ L^{-d(q-1)} & \text{Metal} \end{cases}$$

$$\tau_q = \underset{\text{normal}}{d(q-1)} + \underset{\text{anomalous}}{\Delta_q} \equiv D_q(q-1)$$

# Multifractality of the wave functions $\psi_\alpha(i)$

**Moments of the inverse participation ratio:**

$$I_q(N) \equiv \sum_i |\psi_a(i)|^{2q}$$

$$I_1(N) = 1$$

normalization

**Scaling with $N \to \infty$**

$$I_q(N) = O(N^0) N^{-\tau_q}$$

$$\tau_1 = 0$$

**Ergodicity:**

$$\tau_q = q - 1 \quad \Leftrightarrow \quad |\psi_a(i)|^2 = O(N^{-1})$$

**Exponentially localized states:**

$$\tau_q = 0 \quad \forall q$$

**Multifractality**

$$D_q \equiv \frac{\tau_q}{q-1}$$

Fractal dimensions **differ from** $0$ and $1$ and depend on $q$

# Spectrum of fractal dimensions

Statistics of the onsite values of the eigenfunctions $\left|\psi_a(i)\right|^2$

Distribution function $P(\alpha)$

$$\alpha_i = -\frac{\ln\left|\psi_a(i)\right|^2}{\ln N}$$

random variable

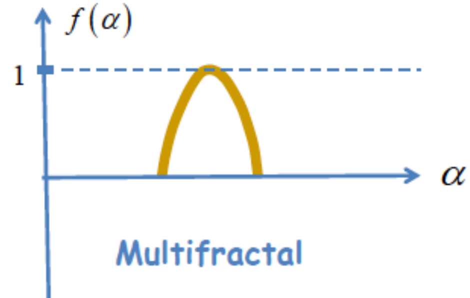
## Typical spectrum of fractal dimensions

$$\boxed{\alpha_q : \left[\frac{\partial f}{\partial \alpha}\right]_{\alpha=\alpha_q} = q}$$

$$\boxed{I_q = N^{-\tau_q}}$$

$$\boxed{\tau_q = q\alpha_q - f(\alpha_q)}$$

$$\boxed{D_q = \frac{q\alpha_q - f(\alpha_q)}{q-1}}$$

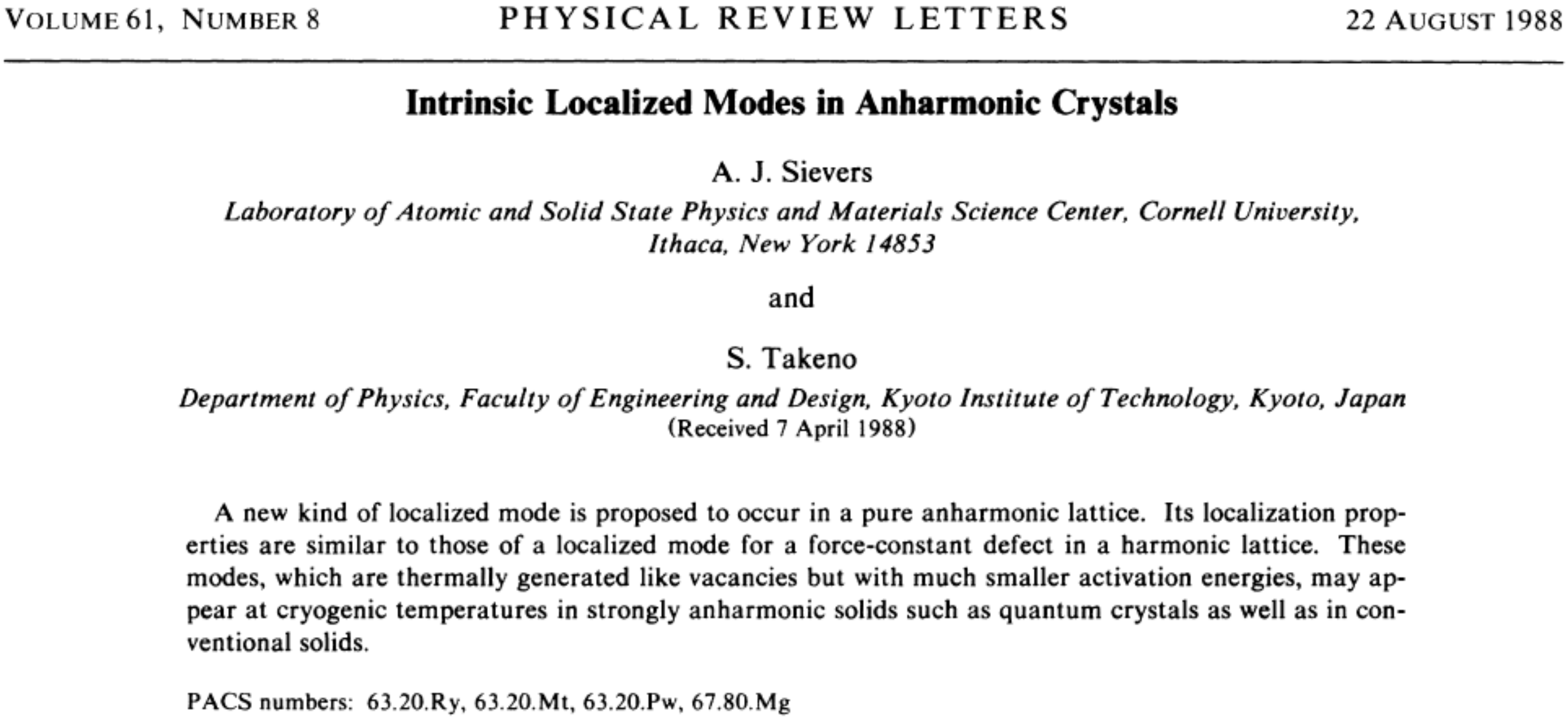


Multifractal

# Intrinsic localized phonon modes at a nonlinear lattice

- ✓ Energy localized vibration in nonlinear lattices is known as intrinsic localized mode.

- ✓ The intrinsic localized mode can move without decaying its energy concentration.



VOLUME 61, NUMBER 8 PHYSICAL REVIEW LETTERS 22 AUGUST 1988

## Intrinsic Localized Modes in Anharmonic Crystals

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(Received 7 April 1988)

A new kind of localized mode is proposed to occur in a pure anharmonic lattice. Its localization properties are similar to those of a localized mode for a force-constant defect in a harmonic lattice. These modes, which are thermally generated like vacancies but with much smaller activation energies, may appear at cryogenic temperatures in strongly anharmonic solids such as quantum crystals as well as in conventional solids.

PACS numbers: 63.20.Ry, 63.20.Mt, 63.20.Pw, 67.80.Mg

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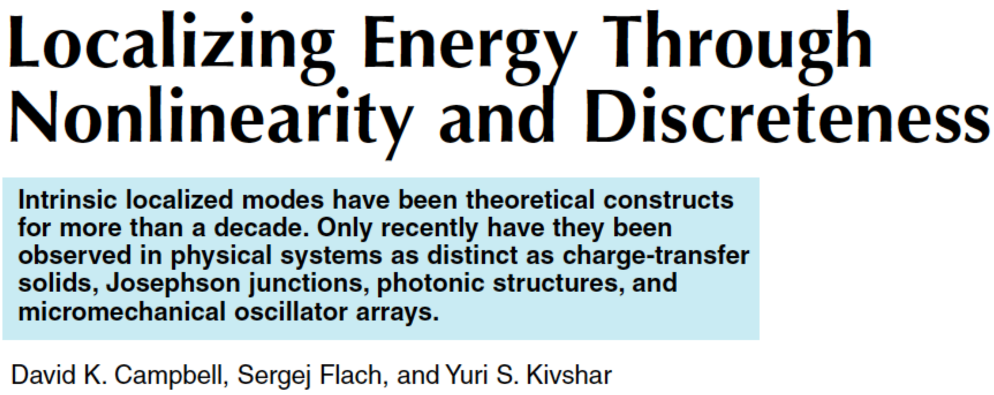


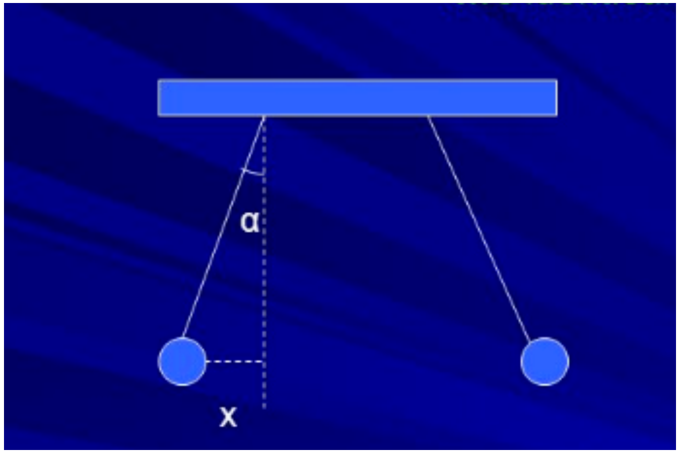
## Localizing Energy Through Nonlinearity and Discreteness

**Intrinsic localized modes have been theoretical constructs for more than a decade. Only recently have they been observed in physical systems as distinct as charge-transfer solids, Josephson junctions, photonic structures, and micromechanical oscillator arrays.**

David K. Campbell, Sergej Flach, and Yuri S. Kivshar

# *Two non-linear oscillators*

$$\omega_i = \omega_i(A_i)$$



$$\omega_2/\omega_1 \neq p/q$$

with p and q
being two co-prime
positive integers.

***Different amplitudes results in different frequencies***

**For strictly incommensurate frequencies, no possible resonances exist between any of the oscillators’ harmonics.**

**The Kolmogorov-Arnold-Moser (KAM) theorem of nonlinear dynamical systems, which establishes that the incommensurate motions do remain rigorously stable for sufficiently weak coupling and ensures that the excitation energy remains localized on the first oscillator.**

$$\frac{d^2\phi_n}{dt^2} - \frac{1}{(\Delta x)^2}\left(\phi_{n+1} + \phi_{n-1} - 2\phi_n\right) - \phi_n + \phi_n^3 = 0.$$

Here $\phi_n(t)$ represents the displacement of a nonlinear "quartic" oscillator at lattice site $n$, so that the equation represents an infinite one-dimensional array of anharmonic oscillators coupled to their nearest neighbors with a coupling strength given by $1/(\Delta x)^2$.

degenerate minima at $\phi_n = \pm 1$

$$\omega_q{}^2 = 2 + (2/\Delta x)^2 \ \sin^2(q/2)$$

$$-\frac{\partial \ln \omega}{\partial \ln \Delta x} = \frac{3A}{\omega}$$

$$2 < \omega_q{}^2 < 2 + (2/\Delta x)^2$$

$$\boxed{\frac{d^2\phi_n}{dt^2}} - \frac{1}{(\Delta x)^2}\left(\phi_{n+1} + \phi_{n-1} - 2\phi_n\right) \boxed{-\phi_n} \boxed{+\phi_n^3} = 0.$$

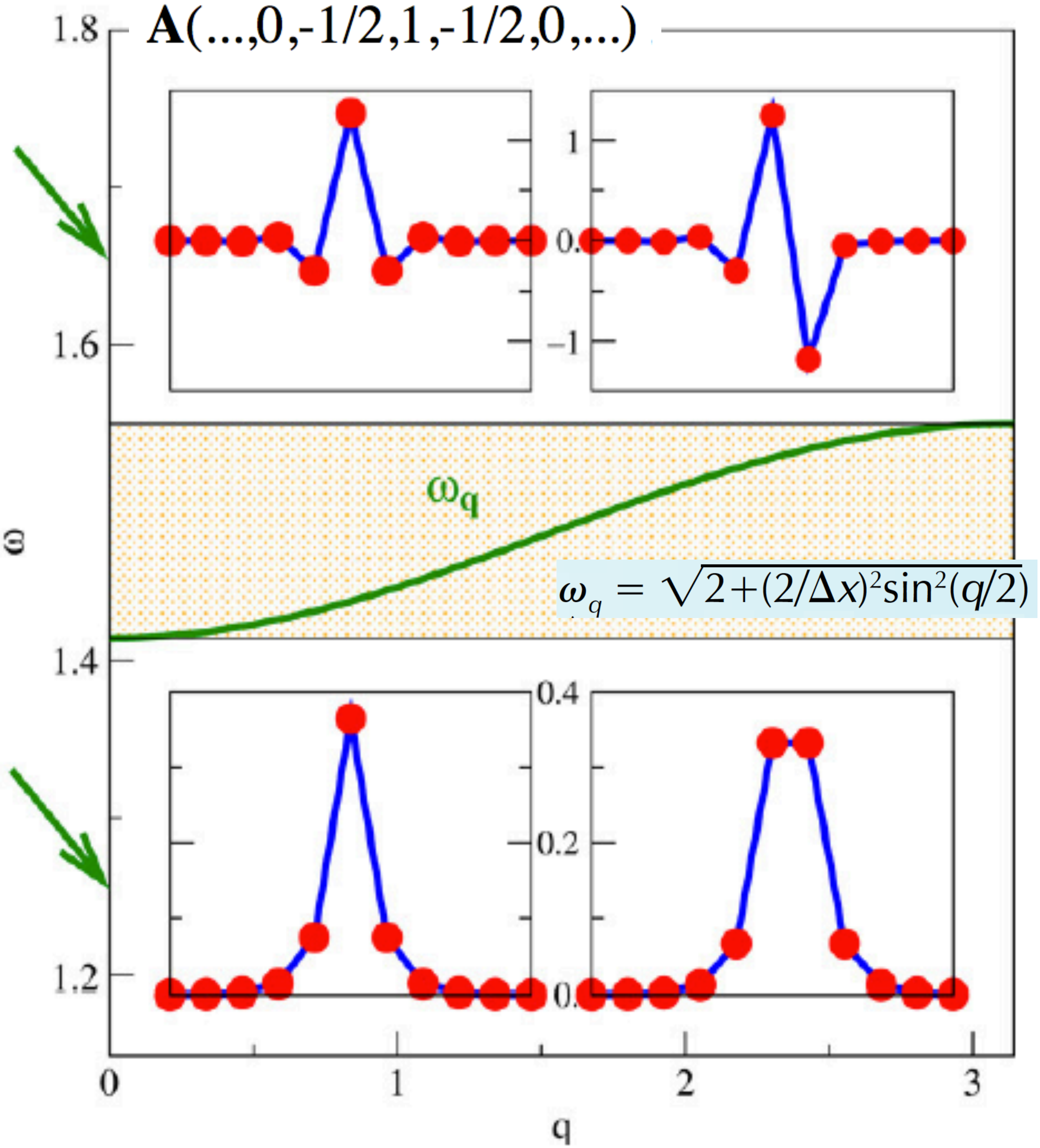
Here $\phi_n(t)$ represents the displacement of a nonlinear "quartic" oscillator at lattice site $n$, so that the equation represents an infinite one-dimensional array of anharmonic oscillators coupled to their nearest neighbors with a coupling strength given by $1/(\Delta x)^2$.

degenerate minima at $\phi_n = \pm 1$

Phonons dispersion relation

$$\omega_q{}^2 = 2 + (2/\Delta x)^2 \ \sin^2(q/2)$$

$$2 < \omega_q{}^2 < 2 + (2/\Delta x)^2$$



**A**(…,0,-1/2,1,-1/2,0,…)

$$\omega_q = \sqrt{2+(2/\Delta x)^2\sin^2(q/2)}$$

Intrinsic localized modes (ILMs), also known as discrete breathers (DBs), are, in fact, typical excitations in perfectly periodic but strongly nonlinear systems.

Hence, there will be no possibility of a linear coupling to the extended modes, even in the limit of an infinite system when the spectrum ω_q becomes dense. This means that the ILM cannot decay by emitting linear waves (that is, phonons) and is hence linearly stable. (1D,2D and 3D)

Discrete intrinsic localized modes in a microelectromechanical resonator
https://arxiv.org/abs/1610.01370

Using the combination of Laser Doppler Vibrometry (LDV) and piezoelectric driving, they observe the two-dimensional transport of ILMs on mechanical structure.

Theoretical studies by Ding Chen (Saclay) and his collaborators gave an explicit algorithm for moving ILMs along the lattice, and calculations of Michel Peyrard (ENS, Lyon) established that ILMs can be generated from thermal fluctuations.

This intuitive understandingof the origin of ILMs/DBs in discrete nonlinear systems was presented in the pioneering paper of Albert Sievers and Shozo Takeno in 1988.

Robert MacKay and Serge Aubry, for example, rigorously proved the existence of DBs in networks of weakly coupled anharmonic oscillators.

The Chen and Peyrard results suggest that ILMs may play critical roles in the transport of energy and other dynamical properties of nonlinear discrete systems, such as melting transitions in solids and folding in polypeptide chains.

The conformational changes and buckling of long biopolymer molecules may occur in response to the excitation of nonlinear localized modes.

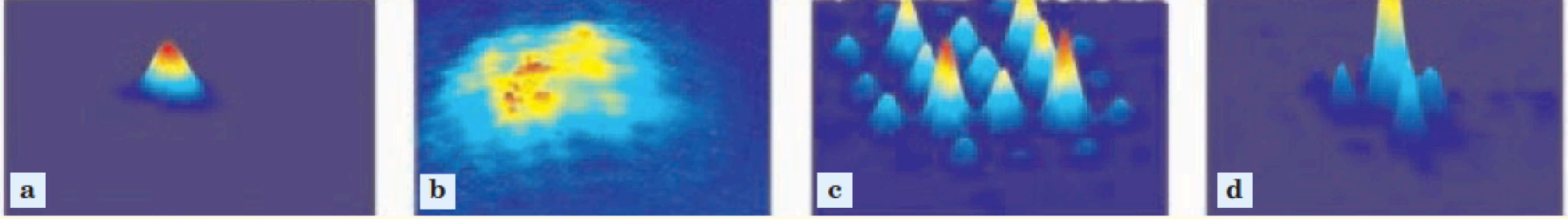


**Figure 4. A two-dimensional intrinsic localized mode** forms in a photonic lattice that was created by optical induction in a photorefractive crystal. A second laser beam provides the input, which is centered on a single site in the photonic lattice. The 3D perspectives show **(a)** the input intensity; **(b)** the linear diffraction output that occurs in the absence of a photonic lattice; **(c)** the discrete linear diffraction, induced by the photonic lattice for weak nonlinearity; and **(d)** an ILM that occurs at larger nonlinearity. (Figure adapted from H. Martin et al., ref. 12.)

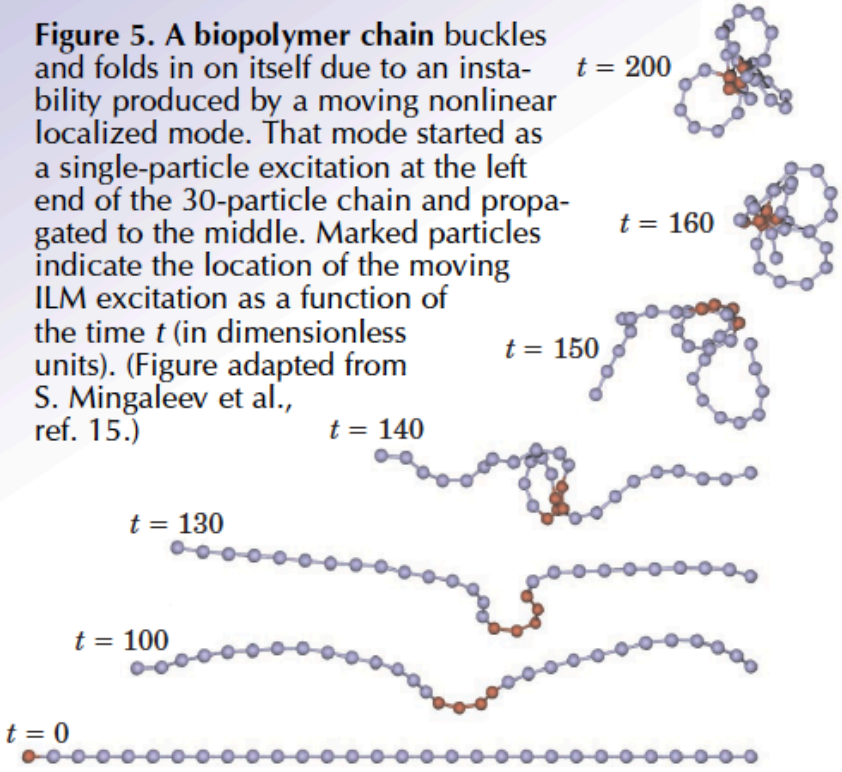


**Figure 5. A biopolymer chain** buckles and folds in on itself due to an instability produced by a moving nonlinear localized mode. That mode started as a single-particle excitation at the left end of the 30-particle chain and propagated to the middle. Marked particles indicate the location of the moving ILM excitation as a function of the time *t* (in dimensionless units). (Figure adapted from S. Mingaleev et al., ref. 15.)

- A.S. Dolgov, Sov. Phys. Solid State 28, 907 (1986).
- A.J. Sievers and S. Takeno, Phy. Rev. Lett. 61, 970 (1988).
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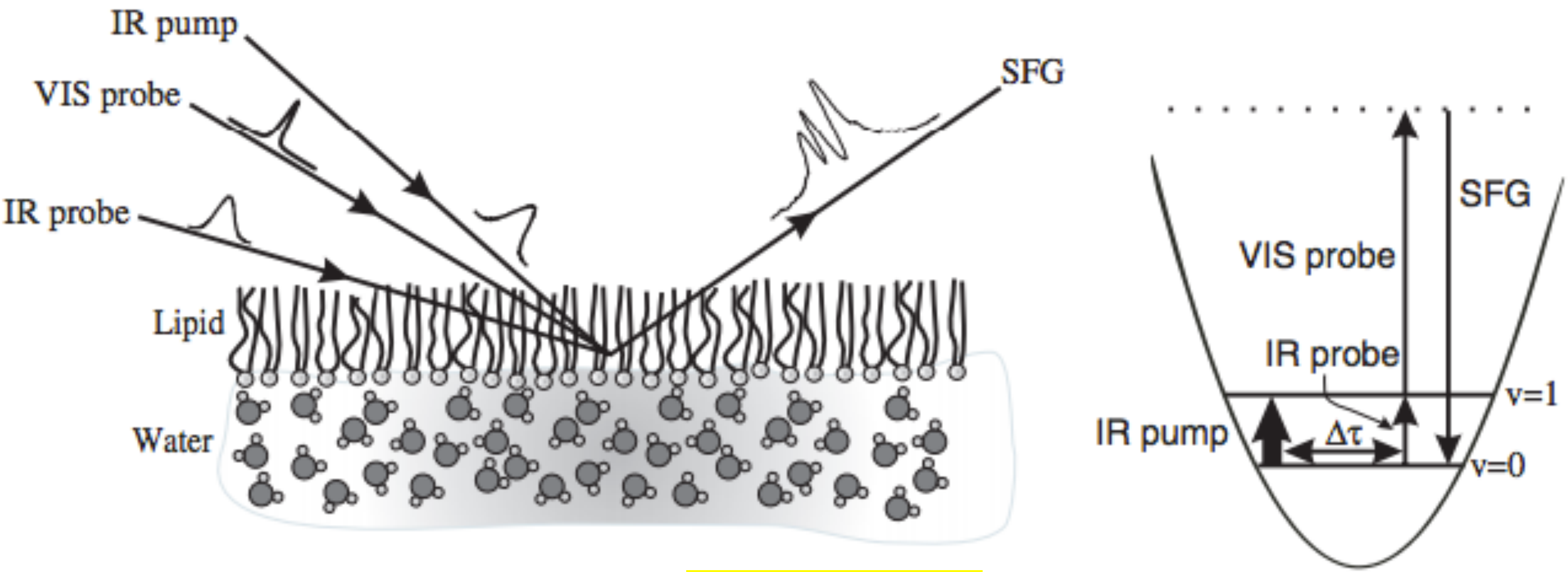
- D. K. Campbell, et al. Localizing Energy Through Nonlinearity and Discreteness, Physics Today 57 , 1, 43 (2004).
- Discrete breathers — Advances in theory and applications

S. Flach and A.V. Gorbach, Physics Reports 467 (2008) 1–116.

# Phonon life time (EXP.)

The time-resolved sum frequency generation (TR-SFG)

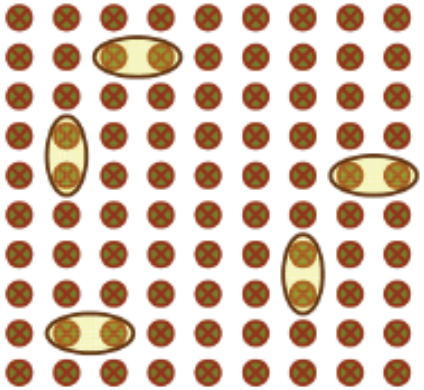


$\tau \sim 1ps$

M. Smits 2007 New J. Phys. 9 390

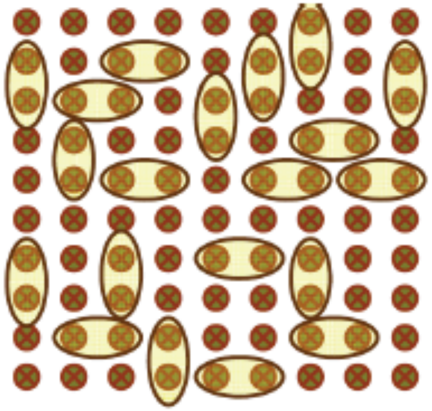
http://web.vu.lt/ff/m.vengris/images/TR_spectroscopy02.pdf

# Thanks



Anderson insulator

Few isolated resonances



Anderson metal

There are many resonances and they overlap

**Transition:** Typically each site is in the resonance with some other one

## Condition for Localization:

$$I < \frac{\text{energy mismatch}}{\text{\# of n.neighbors}}$$

$$\text{energy mismatch} = \left|\varepsilon_i - \varepsilon_j\right|_{typ} = W$$

$$\text{\# of nearest neighbors} = 2d$$

## A bit more precise:

$$\boxed{\frac{I_c}{W} \simeq \left(\frac{1}{2d}\right)\left(\frac{1}{\ln d}\right)}$$

Logarithm is due to the resonances, which are not nearest neighbors

# Condition for Localization:

$$\boxed{\frac{I_c}{W} \simeq \left(\frac{1}{2d}\right)\left(\frac{1}{\ln d}\right)}$$

**Q:** Is it correct?

**A1:** For low dimensions – NO. $I_c = \infty$ for $d = 1,2$
All states are localized. Reason – loop trajectories

**A2:** Works better for larger dimensions $d > 2$

**A3:** Is exact on the Cayley tree (Bethe lattice)

$$I_c = \frac{W}{K \ln K},$$

$K$ is the branching number